J. Fluid Mech. (1986), *vol.* 165, *pp.* 445–474
Printed in Great Britain

On the Boussinesq model for two-dimensional wave motions in heterogeneous fluids

By T. BROOKE BENJAMIN
Mathematical Institute, 24/29 St Giles, Oxford OX1 3LB

(Received 7 December 1984 and in revised form 30 September 1985)

In the Boussinesq model, which is a standard frame for the analysis of internal-wave phenomena, the fluid has variable density but is incompressible, inviscid and non-diffusive. Without further approximations, which will not be made here, the dynamical equations are nonlinear and the evolutionary problem posed cannot be solved explicitly except by numerical means; but various interesting properties are accessible. In §1, where a previous summary account is recalled (Benjamin 1984), the model is reformulated as a system of integro-differential equations in which the dependent variables are density ρ and density-weighted vorticity σ. The aim subsequently is to survey the model's mathematical consequences in general rather than to examine particular solutions. Very few exact solutions are yet known although approximate solutions are on record describing solitary and periodic waves of permanent form.

In §2 the Hamiltonian representation of the two-component system is noted, being the key to much of the analysis that follows. The complete symmetry group for this system is given in §3. It is composed of nine one-parameter subgroups which are listed first in Theorem 1; then their collective significance in relation to Hamiltonian structure is discussed. In §4 two theorems are given specifying necessary and sufficient conditions for a scalar function to be a conserved density for solutions of the Boussinesq system. There are found to be basically eight such conserved densities which are listed in Theorem 4; and the corresponding conservation laws in integral form for motions between horizontal planes are stated in Theorem 5.

The meaning of impulse according to the Boussinesq model is examined in §5. The two linear components of impulse density and the density of impulsive couple are revealed by the preceding examination of symmetries and local conservation laws; but care is needed to identify physical interpretations of the integral conservation laws that involve impulse. Two laws relating impulse to kinematic properties of the density distribution are particularly strange. Separate treatments are needed for the cases where the fluid-filled domain D is the whole of $\mathbb{R}^2$, where D is a half-space with rigid horizontal boundary (which case is in several respects the most delicate) and where D is a horizontal infinite strip. Finally, in §6, a variational characterization of steady wave motions is explained as a concomitant of Hamiltonian structure, and its implications concerning the stability properties of such motions are reviewed.

Appendix A notes a semi-Lagrangian formulation which has a simpler Hamiltonian structure but a narrower range of application. Appendix B outlines an alternative confirmation of Hamiltonian properties by use of a lemma due to Olver (1980*b*).

1. Introduction

The Boussinesq model is a mathematical prototype which simulates the main attributes of internal-wave phenomena in the atmosphere and oceans. The fluid has variable density but is taken to be strictly incompressible, non-diffusive and inviscid. So the model bypasses thermodynamic and diffusive effects that determine the basic density structure in the geophysical situations simulated. Density variations make their effect primarily through buoyancy, and another approximation also commonly named after Boussinesq is to ignore them in accounting for fluid inertia. In the Eulerian equations of motion density ρ is then treated as a constant except where it is multiplied by the gravitational acceleration g. This approximation is deceptive, however, leading for example to qualitatively false predictions about solitary waves in a stratified fluid between horizontal planes (see Benjamin 1966, Appendix), and it will not be made in the following analysis.

For planar motions, which are the only ones to be treated here, the time-dependent Boussinesq model is still formidably intricate and the progress of knowledge about it has been sporadic over many decades. Notably, Seliger & Whitham (1968, p. 8) discovered an ingenious variational principle for it in terms of Clebsch variables, and they among many other writers commented on the mathematical peculiarity of the model. A new approach has been summarized in a recent paper (Benjamin 1984, §5.3; henceforth to be denoted B), which took the Boussinesq model as one of several examples meant to demonstrate the advantages of finding Hamiltonian representations of hydrodynamic problems. To be reinforced by the discussion that follows, this approach appears to give the most robust overall view of the model's properties, in particular identifying in a natural way the appropriate definition of impulse for it and framing all its conservation laws.

Eight such laws will be given in the present account, most of them for the first time, and the opportunity is taken for additional commentary on the material of B. The very delicate question of impulse in an unbounded Boussinesq fluid will be resolved in §5. Finally, in §6, again adding to the summary account in B, a discussion of steady waves in the Boussinesq model is presented. It will be explained that present means are well suited to proving the stability of solitary waves in stably stratified fluids of great depth, a property strongly suggested by experimental observation but not yet explained exactly; however, the hard calculation needed to complete a proof is not attempted here.

From B let us recall the basic equations. Incompressibility of the fluid implies that $\operatorname{div}(u, v) = 0$, where u and v are its velocity components in the horizontal x-direction and (upward) vertical y-direction respectively. Therefore a stream function $\psi(x, y, t)$ exists such that $u = \psi_y$ and $v = -\psi_x$. The vorticity is $\zeta = v_x - u_y = -\Delta\psi$; and a suitable strategy for dealing with homogeneous incompressible fluids (cf. B, §§5.1 & 5.2) is to take ζ as a Hamiltonian variable and consider ψ as a linear transformation of it. Thus $\psi = (-\Delta)^{-1}\zeta$ with the inverse operator incorporating the boundary conditions imposed on ψ. In contrast a crucial step in B, §5.3, was to take as the first variable the density ρ and as the second

$$\sigma = (\rho v)_x - (\rho u)_y = -(\rho\psi_x)_x - (\rho\psi_y)_y = \mathrm{L}_{(\rho)}\sigma, \tag{1}$$

say. Since ρ remains everywhere positive and bounded, the ρ-dependent linear operator $\mathrm{L}_{(\rho)}$ remains strongly elliptic. Hence, according to the Lax–Milgram theorem as noted in B (p. 35), $\mathrm{L}_{(\rho)}$ has, respective to each possible ρ, a unique inverse incorporating the boundary conditions on ψ. As will be made more explicit below, we thus have in principle

$$\psi = \mathrm{B}_{(\rho)}\sigma, \tag{2}$$

where $\mathrm{B}_{(\rho)} = \mathrm{L}_{(\rho)}^{-1}$ is for each ρ a respective symmetric linear operator generally transforming measurable functions (from $\mathrm{L}^2(D)$, say) into smoother functions. When the fluid-filled domain D is the whole of $\mathbb{R}^2$, one requires that $\psi \to 0$ and $|\nabla\psi| \to 0$ as $r = (x^2+y^2)^{\frac{1}{2}} \to \infty$; and when D is the infinite strip $\mathbb{R} \times (0, h)$, the additional kinematic conditions $v = 0$ for $y = 0$ and $y = h$ are covered by requiring $\psi = 0$ on these two rigid boundaries. With corresponding conditions on ψ the representation (2) also holds when D is a half-space with rigid boundary $y = 0$, also when D is any other open, simply connected subdomain of $\mathbb{R}^2$ with rigid boundaries wherever it is bounded.

[The meaning of $\mathrm{B}_{(\rho)}$ may be clarified as follows. Multiplying (1) by ψ, integrating by parts over D and using the boundary conditions, one obtains a quadratic convex functional in ψ which represents kinetic energy and is continuous in the Sobolev space $H_0^1(D)$. Accordingly, ψ given by (2) as a transformation of σ may be appreciated as the maximizer for the variational principle

$$\int_D \tfrac{1}{2}\sigma\psi \,\mathrm{d}x\,\mathrm{d}y = \max_{\xi} \int_D (\sigma\xi - \tfrac{1}{2}\rho|\nabla\xi|^2)\,\mathrm{d}x\,\mathrm{d}y,$$

which defines the quadratic *polar functional* in σ. Here the competitors ξ for the maximum can range over $H_0^1(D)$, and in fact the maximizer has the attribution $\psi \in H_0^1(D) \cap H^2(D)$ if merely $\sigma \in \mathrm{L}^2(D)$. Further regularity of ψ follows correspondingly from that of σ. The existence of $\psi = \mathrm{B}_{(\rho)}\sigma$ is ensured by the varied functional on the right being upper semi-continuous with respect to weak convergence in $H_0^1(D)$, and its uniqueness is ensured by this functional being anti-convex. Note that whereas the boundedness and positivity of ρ are essential to this recipe, ρ does not need to be continuous.]

The equation of mass conservation satisfied everywhere in D can be written

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \rho_t + \partial(\rho, \psi) = 0, \tag{3}$$

with $\partial(.,.)$ denoting the Jacobian derivative $\partial(.,.)/\partial(x,y)$. And by elimination of pressure from Euler's equations of motion the equation for σ is found to be

$$\sigma_t + \partial(\sigma, \psi) + \partial(\rho, gy - \tfrac{1}{2}|\nabla\psi|^2) = 0, \tag{4}$$

in which the first two terms on the left represent $\mathrm{D}\sigma/\mathrm{D}t$. Equations (3) and (4) have just two dependent variables, ρ and σ, because ψ is for each t the transformation (2) of them *qua* functions of x, y. Together with initial data $\rho(x,y,0)$ and $\sigma(x,y,0)$, these equations compose a Cauchy problem that fully determines the motion for $t > 0$. In common with standard vorticity equations, this formulation circumvents the awkward dependence of the original dynamical equations on pressure (cf. B, p. 59).

Note that (3) and (4) are both in conservation form since the operation $\partial(.,.)$ evidently generates a divergence. Equation (3) implies ρ to evolve in the set of *rearrangements* of the bounded positive function $\rho(x,y,0)$ (Hardy, Littlewood & Pólya 1964, chapter 10), which set has many interesting properties. For instance, if the measure of D is finite (a necessary qualification since the support of ρ fills D), the set is strongly closed but not convex and so not weakly closed in any of the spaces $\mathrm{L}^p(D)$ $(p \geqslant 1)$. Then for any measurable function of ρ its integral over D is a constant of the motion. Cases where the measure of D is infinite are of greater interest at present, and then a representative invariant integral associated with (3) is

$$I_1 = \int_D (\rho - \rho_0)\, \mathrm{d}x\, \mathrm{d}y, \tag{5}$$

where $\rho_0 = \rho_0(y)$ is density in the basic quiescent state of the system. (This is the first in a hierarchy of integrals I_j $(j = 1, \ldots, 8)$ appearing in statements of conservation laws.) The integral (5) can be supposed to converge in the case of a localized motion, likewise if the integral is over one period in the case of an x-periodic motion; and in either case I_1 is easily confirmed from (3) to be a constant of the motion. Note that I_1 is not necessarily zero: allowance is made for the possibility of fluid having been added to or subtracted from the basic state.

By use of the stated boundary conditions and the assumption that density is constant along horizontal plane boundaries, it is easy to confirm from (4) that

$$I_2 = \int_D \sigma\, \mathrm{d}x\, \mathrm{d}y \tag{6}$$

is another constant of the motion in the cases that D is unbounded or is bounded by horizontal planes. But $\sigma(x, y, t)$ is generally not a rearrangement of $\sigma(x, y, 0)$. A characterization of the configuration space in which σ evolves will be noted incidentally in §6. The invariants (5) and (6) are considered as being representative, but in fact they exemplify a more general conservation law to be noted in §4.

In the present discussion attention will be limited to motions that disturb a state of rest. For more general Boussinesq systems in which the quiescent state features a non-uniform horizontal current $U(y)$, a Hamiltonian representation was also identified in the previous paper (B, pp. 37 & 38); and virtually all the ideas to be examined here admit extension to this more intricate case. The dependent variables for it are again ρ and σ, the latter representing an addition to the (time-dependent) quantity $-(\rho U)_y$ which also arises from the first identity in (1), but the Hamiltonian structure is subtly modified. Remarkably, the components of impulse in such systems remain as defined in §§4 and 5 below.

In Appendix A a simpler Hamiltonian structure will be noted to hold under the somewhat restrictive assumption that, for each x, the heights of isopycnic surfaces continue to be disposed in the same order as the basic state. In terms of semi-Lagrangian variables, the equations of motion can then be expressed in canonical (Darboux) form. But the needed relation between the stream function and the new dependent variables is more complicated than (1).

The material of this paper, particularly of §6, has much in common with that of a recent paper by Abarbanel *et al.* (1986; see also Marsden 1976), who examine at length the Hamiltonian structures of both two-dimensional and three-dimensional flow problems for incompressible stratified fluids. Their findings are directed mainly towards questions about the stability of steady flows. Although the 'Boussinesq approximation' noted in our opening paragraph to be avoided here is adopted by them, their longer account is mathematically more complete than the present one, whose principal aims are different. Another noteworthy precedent, again with different aims and style of treatment, is a paper by Ripa (1981) on symmetries and conservation laws for internal waves.

2. Hamiltonian representation

Relative to a stable state of rest in which the mass density is a non-increasing function ρ_0 of y alone, the total energy $\hat{H}$ of the motion is the integral of

$$H = \tfrac{1}{2}\rho|\nabla\psi|^2 + gy(\rho-\rho_0) \tag{7}$$

over D. Correspondingly, the first (infinitesimal) variation of $\hat{H}$ is the integral of

$$\begin{aligned}\dot{H} &= \dot{\rho}(\tfrac{1}{2}|\nabla\psi|^2 + gy) + \rho\nabla\psi\cdot\nabla\dot{\psi} \\ &\sim \dot{\rho}(gy - \tfrac{1}{2}|\nabla\psi|^2) - \dot{\psi}\nabla\cdot(\rho\nabla\psi + \dot{\rho}\nabla\psi).\end{aligned}$$

The equivalence $\sim$ means equality except for a divergence which makes no contribution to the variation of $\hat{H}$, since by its definition in (2) ψ vanishes on the boundaries and at infinity for all ρ and σ (cf. B, p. 35). The second group of terms on the right is the same as $\psi\dot{\sigma}$. Thus the Euler derivatives of H (i.e. the variational derivatives of $\hat{H}$) are seen to be

$$\mathscr{E}_\rho H = gy - \tfrac{1}{2}|\nabla\psi|^2, \quad \mathscr{E}_\sigma H = \psi = \mathrm{B}_{(\rho)}\sigma. \tag{8}$$

The second result is otherwise evident because $\frac{1}{2}\rho|\nabla\psi|^2 \sim \frac{1}{2}\sigma\mathrm{B}_{(\rho)}\sigma$ and for fixed ρ the operator $\mathrm{B}_{(\rho)}$ is symmetric in $\mathrm{L}^2(D)$.†

Let the solution of (3) and (4) be expressed as the column vector $\boldsymbol{\omega} = [\rho, \sigma]^{\mathrm{T}}$; and write $\mathscr{E}H$ for $[\mathscr{E}_\rho H, \mathscr{E}_\sigma H]^{\mathrm{T}}$. A comparison with (8) then shows (3) and (4) to have the (generalized) Hamiltonian representation

$$\boldsymbol{\omega}_t = \mathbf{J}\mathscr{E}H, \tag{9}$$

in which the cosymplectic (Hamiltonian) operator is

$$\mathbf{J} = \mathbf{J}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\partial(\rho,.) \\ -\partial(\rho,.) & -\partial(\sigma,.) \end{pmatrix}. \tag{10}$$

This $\boldsymbol{\omega}$-dependent matrix of differential operators is skew symmetric in the sense that, for any pair of functions $\boldsymbol{F}$ and $\boldsymbol{G} \in C^1(D \to \mathbb{R}^2)$, the sum $\boldsymbol{F}\cdot\mathbf{J}\boldsymbol{G} + \boldsymbol{G}\cdot\mathbf{J}\boldsymbol{F}$ equals a divergence.‡ To establish that (9) is properly a Hamiltonian system, having the standard properties of such systems, a closure condition needs to be confirmed. This matter can be left aside here, but the needed confirmation will be supplied in Appendix B. A few other, more immediately relevant facts about the underlying Hamiltonian structure will be noted in §3.3, and they amount virtually to an alternative confirmation.

3. Symmetries

The complete symmetry group for the system of equations (9) has been found to consist of nine one-parameter subgroups. These symmetries can be exposed by a systematic calculation of their infinitesimal generators (cf. Olver 1979, 1980*a*, 1983; Benjamin & Olver 1982, §4), which approach gives assurance that the complete group

† Note that here the definition of Euler derivatives, familiar from the calculus of variations, is conveniently extended to terms depending on non-local (pseudo-differential) operators (cf. B, p. 7, §3 and §6). Being very helpful in the interests of concision, this generalization is legitimate on the understanding that, for example, $\mathscr{E}_\sigma F = \delta\hat{F}/\delta\sigma$, where F is the density whose integral over D, recovers the functional $\hat{F}$ whose variational derivative $\delta/\delta\sigma$ is represented.

‡ Specifically, $\{\rho_y(F_1G_2+F_2G_1)+\sigma_yF_2G_2\}_x - \{\rho_x(F_1G_2+F_2G_1)+\sigma_xF_2G_2\}_y$.

has been identified. Such a calculation will be outlined below merely to the extent that it illuminates the connection with conservation laws. All the results, to be presented first as a theorem, can be confirmed directly by substitution in (9).

In the theorem the infinitesimal generator of each symmetry is recorded in its 'standard form' $\boldsymbol{P}$ (cf. Olver 1980*b*, §5) which will be useful in what follows. The main point about $\boldsymbol{P}$, a two-component function of $x, y, t, \boldsymbol{\omega}$ and derivatives, is that if $\boldsymbol{\omega}(x, y, t)$ is any solution of (9), a continuous family of other solutions $\overline{\boldsymbol{\omega}}(x, y, t; \epsilon)$ parametrized by *infinitesimal* $\epsilon \in R$ is obtained by solving

$$\overline{\boldsymbol{\omega}}_\epsilon = \boldsymbol{P}, \quad \overline{\boldsymbol{\omega}}(x, y, t; 0) = \boldsymbol{\omega}(x, y, t). \tag{11}$$

The family of new solutions can be extended to all $\epsilon \in R$ by exponentiation of the infinitesimal generator – or simply by inspection after solving (11) and then checking the estimate by substitution in (9). Families of new solutions thus obtained are recorded in the theorem for all nine subgroups. For the first six examples ϵ denotes each additive group parameter; and for each of the last three $\lambda = e^\epsilon > 0$ denotes a multiplicative parameter.

3.1. *List of symmetries*

The symmetries are numbered 3 to 11 because the first six of them will presently be linked to conservation laws which add to the two already noted in §1. Physical interpretations associable with the various symmetry groups are also indicated (cf. Benjamin & Olver 1982, pp. 148–150).

Theorem 1. *The symmetry group for the two-dimensional Boussinesq system has nine one-parameter subgroups G_j ($j = 3, \dots, 11$) with the following infinitesimal generators:*

Time translation

$$\boldsymbol{P}_3 = -\boldsymbol{\omega}_t;$$

Horizontal translation

$$\boldsymbol{P}_4 = -\boldsymbol{\omega}_x;$$

Vertical translation

$$\boldsymbol{P}_5 = -\boldsymbol{\omega}_y;$$

Horizontal Galilean boost

$$\boldsymbol{P}_6 = t\boldsymbol{\omega}_x + [0, \rho_y]^{\mathrm{T}};$$

Vertical Galilean boost

$$\boldsymbol{P}_7 = t\boldsymbol{\omega}_y - [0, \rho_x]^{\mathrm{T}};$$

Gravity-compensated rotation

$$\boldsymbol{P}_8 = (y + \tfrac{1}{2}gt^2)\,\boldsymbol{\omega}_x - x\boldsymbol{\omega}_y + [0, gt\rho_y]^{\mathrm{T}};$$

Vertical acceleration

$$\boldsymbol{P}_9 = t\boldsymbol{\omega}_t - gt^2\boldsymbol{\omega}_y + [0, \sigma + 2gt\rho_x]^{\mathrm{T}};$$

Trivial scaling

$$\boldsymbol{P}_{10} = \boldsymbol{\omega};$$

Scaling

$$\boldsymbol{P}_{11} = x\boldsymbol{\omega}_x + y\boldsymbol{\omega}_y + \tfrac{1}{2}t\boldsymbol{\omega}_t + [0, \tfrac{1}{2}\sigma]^{\mathrm{T}}.$$

New solutions obtained by transforming any given solution $\boldsymbol{\omega}(x,y,t)$ with $\psi(x,y,t) = \mathrm{B}_{(\rho)}\sigma$ are as follows:

G_3: $\boldsymbol{\omega}(x,y,t-\epsilon)$, $\psi(x,y,t-\epsilon)$;

G_4: $\boldsymbol{\omega}(x-\epsilon,y,t)$, $\psi(x-\epsilon,y,t)$;

G_5: $\boldsymbol{\omega}(x,y-\epsilon,t)$, $\psi(x,y-\epsilon,t)$;

G_6: $\boldsymbol{\omega}(x+\epsilon t,y,t)+[0, -\epsilon\rho_y(x+\epsilon t,y,t)]^{\mathrm{T}}$, $\psi(x+\epsilon t,y,t)-\epsilon y$;

G_7: $\boldsymbol{\omega}(x,y+\epsilon t,t)+[0, -\epsilon\rho_x(x,y+\epsilon t,t)]^{\mathrm{T}}$, $\psi(x,y+\epsilon t,t)+\epsilon x$;

G_8: $\boldsymbol{\omega}(\tilde{x},\tilde{y},t)+[0, gt\{(1-\cos\epsilon)\rho_{\tilde{x}}(\tilde{x},\tilde{y},t)+\sin\epsilon\,\rho_{\tilde{y}}(\tilde{x},\tilde{y},t)\}]^{\mathrm{T}}$,
$\psi(\tilde{x},\tilde{y},t)-gt\{(1-\cos\epsilon)\tilde{x}+\sin\epsilon\,\tilde{y}\}$,

with $\tilde{x} = x\cos\epsilon+(y+\tfrac{1}{2}gt^2)\sin\epsilon$, $\tilde{y} = -x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\cos\epsilon-\tfrac{1}{2}gt^2$;

G_9: $[\rho(x,y',\lambda t), \lambda\sigma(x,y',\lambda t)-gt(1-\lambda^2)\rho_x(x,y',\lambda t)]^{\mathrm{T}}$,
$\lambda\psi(x,y',t)+gt(1-\lambda^2)\,x$,

with $y' = y+\tfrac{1}{2}gt^2(1-\lambda^2)$;

G_{10}: $\lambda\boldsymbol{\omega}(x,y,t)$, $\psi(x,y,t)$;

G_{11}: $[\rho(\lambda x,\lambda y,\lambda^{\frac{1}{2}}t), \lambda^{\frac{1}{2}}\sigma(\lambda x,\lambda y,\lambda^{\frac{1}{2}}t)]^{\mathrm{T}}$, $\lambda^{-\frac{3}{2}}\psi(\lambda x,\lambda y,\lambda^{\frac{1}{2}}t)$.

It should be acknowledged that the facts here listed are irrespective of boundary and initial conditions. Plainly, such conditions cannot increase the number of symmetries admitted by any particular problem, and the number will usually be reduced a lot. For example, only those labelled 3, 4, 6 and 10 remain symmetry groups when a horizontal rigid boundary is present, and only 3 and 10 when there is a boundary that is other than horizontal. All the symmetries identified are nevertheless interesting as intrinsic properties of the nonlinear equations (9). In particular, those of them that underly conservation laws (see discussion below) constitute physically significant information applicable to every Boussinesq problem, whatever the boundary and initial conditions.

The simple symmetry group generated by $\boldsymbol{P}_5$ is perhaps a little surprising at first sight since H in (9) depends explicitly on y. The latter fact will be reflected in the conservation law linked to G_5, but it does not influence the symmetry. Although y appears in the convenient representation (4) of the equation for σ, the second row of (9), the term so represented is just $g\rho_x$ which, in common with all other terms in (9), is free of explicit dependence on y.

The nine symmetries given here largely match the nine that have been found for the water-wave problem in two space dimensions (Benjamin & Olver 1982, Thm 4.1; Olver 1983). Only the trivial scaling symmetry represented by $\boldsymbol{P}_{10}$ has no obvious counterpart in the other problem, where the remaining symmetry of the nine (G_3 in Benjamin & Olver 1982) is tied to mass conservation and so is comparable with the present, already noted property that ρ is a conserved density. The water-wave problem in a modified Hamiltonian version (B, §6.1) can in fact be recovered as an extreme example of (9): one takes $\rho = 1$ for $y \leqslant \eta(x,t)$, $\rho = 0$ for $y > \eta(x,t)$, and interprets the equations in a distributional sense. But the exercise is complicated and has little interest.

3.2. *Characterization of symmetries*

As Theorem 1 can be verified directly, without recourse to the theory of symmetry groups, a full *a priori* derivation of the $\boldsymbol{P}_j$ would add little to the present account, which concentrates on conservation laws and their interpretation.† A systematic calculation of symmetries is essential to prove that the list of them in Theorem 1 is complete, as seems very likely, but this exacting objective lies beyond present aims. It will suffice to summarize the basic facts about the definition of symmetries and their bearing on the Hamiltonian structure of (9), which facts will illuminate the results to be given in §4.

Consider the infinitesimal generator of a one-parameter symmetry group for (9) to be

$$\alpha \frac{\partial}{\partial x}+\beta \frac{\partial}{\partial y}+\tau \frac{\partial}{\partial t}+\gamma_1 \frac{\partial}{\partial \rho}+\gamma_2 \frac{\partial}{\partial \sigma}, \tag{12}$$

which acts on smooth scalar functions of (x, y, t, ρ, σ) (cf. Olver 1979; 1980*a*, §4.3; Benjamin & Olver 1952, pp. 156 and 181). The 'standard representation' of the generator as explained in the context of (11) is correspondingly

$$\boldsymbol{P} = -\alpha\boldsymbol{\omega}_x - \beta\boldsymbol{\omega}_y - \tau\boldsymbol{\omega}_t + [\gamma_1, \gamma_2]^{\mathrm{T}}. \tag{13}$$

In common with most applications of Lie group theory to physical problems, it appears at present that the coefficients α, β and τ in (12) and (13) depend on x, y, t only. That is, the independent variables are transformed separately from the dependent variables; the symmetry group is then called *projectable* (cf. Olver 1980*a*, §2.4).

Now, the evolutionary equation (9) for $\boldsymbol{\omega}$ is not a p.d.e. since the integral operation $\psi = \mathrm{B}_{(\rho)}\sigma$ is entailed, and this fact evidently colours the meaning of (12) and (13). Allowance for it has to be made in order to expose transformation properties associable with (9) as a Hamiltonian system, and also to use prolongation theory which is the most widely effective method for finding symmetries of p.d.e.s (Olver 1979; 1980*a*, §2.3). Such resources are made applicable, however, by the simple expedient of allowing explicitly in (12) and (13) for the infinitesimal transformations of ψ, which depend of course upon those of ρ and σ. Thus, a term $\Gamma\partial/\partial\psi$ is added to (12), and correspondingly the vector $\boldsymbol{P}$ given by (13) is assigned a third component $[\psi_\epsilon]_{\epsilon=0} = -\alpha\psi_x - \beta\psi_y + \tau\psi_t + \Gamma$, which according to (1) and (11) must satisfy

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\{\rho\boldsymbol{\nabla}(-\alpha\psi_x-\beta\psi_y-\tau\psi_t+\Gamma)\} &= -\alpha\sigma_x-\beta\sigma_y-\tau\sigma_t+\gamma_2 \\ &\quad -\boldsymbol{\nabla}\cdot\{(-\alpha\rho_x-\beta\rho_y-\tau\rho_t+\gamma_1)\,\boldsymbol{\nabla}\psi\}. \end{aligned} \tag{14}$$

Recall that boundary conditions are not in question here: the model is $D = \mathbb{R}^2$. (The forms of Γ respective to the $\boldsymbol{P}_j$ listed in Theorem 1 are evident from the new solutions listed, and are easily checked by (14). They are $\Gamma_3 = \Gamma_4 = \Gamma_5 = \Gamma_{10} = 0$, $\Gamma_6 = -y$, $\Gamma_7 = x$, $\Gamma_8 = -gty$, $\Gamma_9 = \psi - 2gtx$, $\Gamma_{11} = -\frac{3}{2}\psi$.)

Without going into the mathematical generalities that are exemplified in the following steps (e.g. see Olver 1979; 1980*a*, chapters 1 and 2; 1980*b*), let us note the Lie bracket whereby the symmetries of (9) can be defined. As an extended collective notation for the dependent variables, write $\boldsymbol{\omega}^* = [\rho, \sigma, \psi]^{\mathrm{T}}$; and write $\boldsymbol{P}^*$ for the

† Such a derivation was included in the first version of this paper, but the general formulae (16) and (19) below are more interesting than the detailed calculations.

corresponding extension of $\boldsymbol{P}$ including $[\psi_\epsilon]_{\epsilon=0}$ as the third component. Also let $\boldsymbol{Q}$ denote the right side of (9) (i.e. $\boldsymbol{\omega}_t = \boldsymbol{Q}$); and write $\boldsymbol{Q}^* = [Q_1, Q_2, \psi_t]^{\mathrm{T}}$, although not implying that ψ_t has to be evaluated like Q_1 and Q_2 in terms of $\boldsymbol{\omega}$ or $\boldsymbol{\omega}^*$. Next let k be a multi-index labelling all relevant partial derivatives $\omega^*_{i,k}$ $(i = 1,2,3)$ with respect to x, y and t, and let $\mathscr{D}^k$ denote the total derivative corresponding to k (cf. Olver 1980*b*, §1). Then the necessary and sufficient condition for $\boldsymbol{P}^*$ to be a symmetry of (9) is seen to be that the system of equations

$$\begin{aligned}\frac{\partial \boldsymbol{P}^*}{\partial t} &= (\mathscr{D}^k P^*_i)\frac{\partial \boldsymbol{Q}^*}{\partial \omega^*_{i,k}} - (\mathscr{D}^k Q^*_i)\frac{\partial \boldsymbol{P}^*}{\partial \omega^*_{i,k}} \\ &= [\boldsymbol{Q}^*, \boldsymbol{P}^*] \end{aligned} \tag{15}$$

should be satisfied for all $\boldsymbol{\omega}^*$ (cf. Arnold 1978, §39). Here $\partial/\partial t$ on the left refers to *explicit* dependence on t, and summations are implied by the repeated indices $i = 1,2,3$ and k. Furthermore $\omega^*_3 = \psi$ must be determined by ρ and σ according to (1), and P^*_3 by P^*_1 and P^*_2 according to (14).

The general condition (15) provides the basis for systematically calculating the symmetries of (9). Substituting from (13) for $\boldsymbol{P}$ and separating terms in the various derivatives of ρ, σ and ψ, and in combinations of them, one obtains a large number of simple differential equations to be simultaneously satisfied by the coefficient functions α, β, τ, γ_1 and γ_2 (cf. Benjamin & Olver 1982, §4; Olver 1983). The highest derivatives are second-order, arising from terms such as $\rho_y\,\psi_x\,\psi_{xx}$ in Q_2. The relation (14) delimiting P^*_3 essentially restricts the possibilities, but it is found useful that $\gamma_1 = a\rho$, $\gamma_2 = b\sigma + A\rho_x + B\rho_y$ corresponds to $\Gamma = (b-a)\psi - Ax - By$ if a, b, A, B depend only on t. Details of the exercise are not included here, and the nine symmetries detected are presented in normalized form as the first part of Theorem 1. The calculations strongly suggest that there is no other symmetry, but a proof will not be attempted.

It is clear *a priori* that P^*_3 cannot involve ρ, σ or their derivatives explicitly, so depending only on ψ and its derivatives. Thus the third component (the last row) of (15) is in effect redundant, even though summation over $i = 1,2,3$ in the first group terms on the right is essential. Specifically, since the formulation specifies $Q^*_3 = \psi_t$, the third component is just

$$\frac{\partial P^*_3}{\partial t} = \mathscr{D}_t P^*_3 - (\mathscr{D}^k \psi_t)\frac{\partial P^*_3}{\partial \psi_{,k}},$$

where $\mathscr{D}_t$ is the total derivative in t which is basically defined by the sum of the term on the left and the transferred terms on the right. So the third equation in (15) is satisfied trivially. On this understanding about the meaning of the Lie bracket, the condition for $\boldsymbol{P}$ to be a symmetry can be written in the form

$$\frac{\partial \boldsymbol{P}}{\partial t} = [\boldsymbol{Q}, \boldsymbol{P}], \tag{16}$$

which is the standard one applying to p.d.e.s. When $\boldsymbol{P}$ like $\boldsymbol{Q}$ does not depend explicitly on t, the condition (16) means simply that the operations $\boldsymbol{\omega} \to \boldsymbol{Q}(\boldsymbol{\omega})$ and $\boldsymbol{\omega} \to \boldsymbol{P}(\boldsymbol{\omega})$ commute.

3.3. *Relation of symmetries to Hamiltonian structure*

First a Poisson bracket associated with (10) must be noted. It composes a $\boldsymbol{\omega}$-dependent scalar density from any two such densities, say F and G, and is defined only within the equivalence classes ignoring divergencies. It is expressed by

$$\begin{aligned}\{F,G\} &\sim \mathscr{E}F\cdot\mathbf{J}\mathscr{E}G \sim -\{G,F\}\\ &\sim \rho\partial(\mathscr{E}_\rho F,\mathscr{E}_\sigma G)+\rho\partial(\mathscr{E}_\sigma F,\mathscr{E}_\rho G)+\sigma\partial(\mathscr{E}_\sigma F,\mathscr{E}_\sigma G);\end{aligned} \tag{17}$$

and a long but straightforward calculation shows that the bracket satisfies the Jacobi identity

$$\{E,\{F,G\}\}+\{F,\{G,E\}\}+\{G,\{E,F\}\}\sim 0.$$

The calculation has the same pattern as that in Appendix B where the closure condition is confirmed in terms of differential forms, and so in effect it serves as an alternative confirmation.

[In the present uses of (17), the divergences left implicit by $\sim$ generally turn out to have zero integrals over D, as they must to make simple sense of the present treatment. For example, on a rigid horizontal boundary (where $\rho_x = 0$) either $\mathscr{E}_\sigma F$ or $\mathscr{E}_\sigma G$ is zero (cf. second footnote on p. 449). Accordingly, by integration of (17) over D, a Poisson bracket may also be defined as a functional composed from functionals $\hat{F}$ and $\hat{G}$. This view, a somewhat more orthodox one, is taken by Abarbanel *et al.* (1986, equation (7.69)) in covering the same ground as (17) – albeit simplified by the Boussinesq approximation and approached by use of Clebsch variables.]

The most important result concerning Hamiltonian structure can now be stated. It is the equality

$$[\mathbf{J}\mathscr{E}F,\mathbf{J}\mathscr{E}G]=\mathbf{J}\mathscr{E}\{F,G\}, \tag{18}$$

were $[.,.]$ is the Lie bracket used in (16) (cf. Olver 1980*b*, Theorem 4.2). Corresponding results are known from studies of simpler Hamiltonian models, such as systems of o.d.e.s or of p.d.e.s in canonical Hamiltonian form; but the present example is a little unusual in the kind of prolongation that has been needed in defining $[.,.]$. Equation (18) can be verified by a straightforward calculation which involves only differentiations of $\mathscr{E}F$ and $\mathscr{E}G$, whose possible dependence on non-local operations (e.g. when F or $G = H$) does not affect the issue. On the left side of the equation these Euler derivatives are to be treated as functions of ρ, σ and ψ in accord with our definition of the Lie bracket. On the right side considerable simplification results because the Euler derivative of any divergence is zero, while the second line of (17) shows $\{F,G\}$ to be the sum of ρ times a divergence and σ times another. The two sides of the equation are thus found to reduce to

$$-[\partial(\rho,\partial(\mathscr{E}_\sigma F,\mathscr{E}_\sigma G)),\quad \partial(\sigma,\partial(\mathscr{E}_\rho F,\mathscr{E}_\sigma G))+\partial(\rho,\partial(\mathscr{E}_\sigma F,\mathscr{E}_\sigma G)+\partial(\mathscr{E}_\sigma F,\mathscr{E}_\rho G))]^{\mathrm{T}}.$$

Since (18) can be taken as the basic definition of the Hamiltonian property attributed to $\mathbf{J}$ (cf. Manin 1979), this direct proof is yet another alternative to that in Appendix B.

Coming at last to the pivotal property that will be used in §4 to account for conservation laws, we combine (16) and (18) with $\boldsymbol{Q} = \mathbf{J}\mathscr{E}H$ and with $\boldsymbol{P} = \mathbf{J}\mathscr{E}T$, which relates a density T to a symmetry group. There thus follows

$$\frac{\partial \boldsymbol{P}}{\partial t}=\mathbf{J}\mathscr{E}(H_\epsilon)_0, \tag{19}$$

in which $(H_\epsilon)_0 = \{H, T\} = \mathscr{E}H \cdot \boldsymbol{P}$. Note that (19) is guaranteed *only* when $\boldsymbol{P}$ is a symmetry admitting the representation $\boldsymbol{P} = \mathbf{J}\mathscr{E}T$ for some T; it is not true for all the $\boldsymbol{P}_j$ in Theorem 1.

4. Conservation laws

4.1. *The simplest examples*

The general property to be noted first covers the invariance of I_1 and I_2 given by (5) and (6). Consider

$$T_* = e(\rho) - e(\rho_0) + \sigma f(\rho), \tag{20}$$

in which e and f are arbitrary real functions. The term $-e(\rho_0)$ is included to ensure that T_* has a convergent integral $\hat{T}_*$ over an unbounded D. It is seen from (10) that $\mathbf{J}\mathscr{E}T_* = 0$, and a simple calculation shows that the only $\boldsymbol{\omega}$-dependent densities annihilated by the operation $\mathbf{J}\mathscr{E}$ have the form (20). The attribute $\mathbf{J}\mathscr{E}T_* = 0$ implies that T_* is a conserved density, by which we mean that its total derivative in t is a divergence; for

$$\partial_t T_* = \mathscr{E}T_* \cdot \boldsymbol{\omega}_t = T_* \cdot \mathbf{J}\mathscr{E}H \sim -\mathscr{E}H \cdot \mathbf{J}\mathscr{E}T_* = 0.$$

Hence, taking account of the implicit divergence and finding its integral over D to be zero, provided $\rho_x = 0$ along the one or two horizontal planes that may bound D, we conclude that

$$\frac{\mathrm{d}\hat{T}_*}{\mathrm{d}t} = 0. \tag{21}$$

The properties $\mathrm{d}I_1/\mathrm{d}t = 0$ and $\mathrm{d}I_2/\mathrm{d}t = 0$, which (21) evidently includes, are given prominence as representative ones in the present account. They are considered the most interesting particular cases of (21). But $\hat{I}_*$ in its general form may have importance for some purposes. The general property (21) survives the Boussinesq approximation, whereby σ is replaced by a constant mass density times $\zeta = -\Delta\psi$, and in this light it is discussed by Abarbanel *et al.* (1986, §§2 and 7C).

The facts about T_* are consistent with the characterization (19) of symmetries related to a density T, but they tell us nothing in this regard since the vacuous case $\boldsymbol{P} = 0$ is given. They reflect rather the degeneracy (non-invertibility) of the Hamiltonian operator $\mathbf{J}$, whose null space is precisely the set of all ω-dependent two-vectors $\mathscr{E}T_* = [e'(\rho) + \sigma f'(\rho), f(\rho)]^{\mathrm{T}}$.

4.2. *Adaptation of Noether's theorem*

A valuable purpose served by identifying the Hamiltonian structure of a dynamical problem is to frame a systematic relation between symmetries and conservation laws, which relation is usually accessible through Noether's theorem or some appropriate elaboration of it. A version of the theorem applying to Hamiltonian evolutionary equations with an arbitrary operator $\mathbf{J}$ as in (9) has been given by Olver (1980*b*, §5), being based on the theory of differential forms. His account emphasizes and fully explains that not every symmetry, particularly not a scaling symmetry, is necessarily linked to a conservation law; and his main result (his Theorem 5.2) is close in gist, although not quite suited to, the needs of the present problem. Based on the results established in §§3.2 and 3.3, however, a wholly adequate analogue of Noether's theorem is provided by the following two theorems. They demonstrate first a necessary condition and then sufficient conditions for a scalar function $T(x, y, t, \boldsymbol{\omega})$ to be conserved density. The indicated dependence of T is meant to include dependence on x, y, t explicitly, on $\boldsymbol{\omega}$ and its partial derivatives and also on non-local transformations of $\boldsymbol{\omega}$.

Theorem 2. *Let the scalar function* $T(x, y, t, \boldsymbol{\omega})$ *be a conserved density for solutions of* (9). *Then the two-component function*

$$\boldsymbol{P} = \mathbf{J}\mathscr{E}T \tag{22}$$

represents the infinitesimal generator of a symmetry group for (9).

Proof. The assumption about T means that

$$0 \sim \frac{\partial T}{\partial t} + \mathscr{E}T\cdot\boldsymbol{\omega}_t = \frac{\partial T}{\partial t} + \{T, H\},$$

where $\partial T/\partial t$ refers to explicit dependence on t. The skew symmetry of $\mathbf{J}$ hence implies

$$\frac{\partial T}{\partial t} \sim \{H, T\}, \tag{23}$$

whereupon the operation $\mathbf{J}\mathscr{E}$, which commutes with ∂_t in its present sense, gives

$$\frac{\partial \boldsymbol{P}}{\partial t} = \mathbf{J}\mathscr{E}\{H, T\}.$$

This equation reproduces the characterization of symmetries that was expressed by (19), thus showing $\boldsymbol{P}$ to represent a symmetry group. □

Because $\mathbf{J}$ has a non-trivial null space, as appreciated in §4.1, the condition (22) is not sufficient for T to be conserved density. Specifically, T may include essential terms that depend explicitly on t but are annihilated by $\mathbf{J}\mathscr{E}$, so being unrepresented in (22). Sufficient conditions are specified as follows.

Theorem 3. *Let* $T(x, y, t, \boldsymbol{\omega})$ *satisfy*

$$\mathbf{J}\mathscr{E}T = \boldsymbol{P}, \tag{22'}$$

where $\boldsymbol{P}$ *represents a symmetry of* (9), *and also satisfy*

$$\mathscr{E}_\rho(\partial T/\partial t) = g\beta, \tag{24}$$

where β *is the coefficient of* $-\omega_y$ *in* $\boldsymbol{P}$ *and* $\beta_y = 0$. *Then* T *is a conserved density for solutions of* (9).

Proof. The fact that $\boldsymbol{P}$ given by (22′) represents a symmetry implies (19), from which (23) now needs to be inferred. Writing out $\{H, T\} = \mathscr{E}H\cdot\boldsymbol{P}$ in full upon substitution from (8) and (13), one sees that the only component whose Euler derivative may lie non-trivially in the null space of $\mathbf{J}$ is $-gy\beta\rho_y$. Since $\mathscr{E}_\rho(-gy\beta\rho_y) = g(\beta + y\beta_y)$, this term makes no contribution to (19) when $\beta_y = 0$. (The possibility of another such term arising from $gy\gamma_1$ needs consideration but is easily dismissed upon trial of the requisite γ_1 in (15), which must hold for all ρ and σ.) Since all remaining terms in (23) are guaranteed by (19) in the light of (22′), it follows that (24) is a sufficient extra condition to guarantee (23). Therefore, using $\{H, T\} \sim -\{T, H\}$, we infer

$$\frac{\partial T}{\partial t} + \{T, H\} \sim 0,$$

which confirms T to be a conserved density. □

Theorem 2 is valuable in showing that not all of the symmetries listed in Theorem 1 can be tied to ordinary conservation laws. It appears that none of the scaling symmetries $\boldsymbol{P}_9 - \boldsymbol{P}_{12}$ can be represented in the form (22); nor can any linear combination of them. In particular, the component $\frac{1}{2}\sigma$ in the second row of $\boldsymbol{P}_{11}$ and $\frac{1}{2}\boldsymbol{P}_9$ lies in the range of $\mathbf{J}$ only in the nugatory case $\Delta\psi = 0$; and the components of $\boldsymbol{P}_{11} - \frac{1}{2}\boldsymbol{P}_9$ are not divergences. Although the components of $\boldsymbol{P}_{11} - \frac{1}{2}\boldsymbol{P}_9 + 2\boldsymbol{P}_{10}$ are divergences, they are still not in the range of $\mathbf{J}$. These scaling symmetries have significance in relation to families of solutions, but this aspect is passed over here.

Functions $T_j(x, y, t, \omega)$ shown by Theorem 3 to be conserved densities are listed as follows (cf. Benjamin & Olver 1982, theorem 5.2). The third to eighth T_j are numbered respective to the $\boldsymbol{P}_j$ entailed in the conditions (22′) and (24).

Theorem 4. *The two-dimensional Boussinesq system* (9) *has the following eight conserved densities*:

$$\begin{aligned}
&T_1 = \rho-\rho_0, \quad T_2 = \sigma,\\
&T_3 = H \text{ [cf. (7)]}, \quad T_4 = y\sigma,\\
&T_5 = -x\sigma+gt(\rho-\rho_0) = -x\sigma+gtT_1,\\
&T_6 = x(\rho-\rho_0)-ty\sigma = xT_1-tT_4,\\
&T_7 = (y-\tfrac{1}{2}gt^2)(\rho-\rho_0)+tx\sigma = (y+gt^2)\,T_1-tT_5,\\
&T_8 = -\tfrac{1}{2}(x^2+y^2)\,\sigma+gtx(\rho-\rho_0)-\tfrac{1}{2}gt^2y\sigma\\
&\quad\; = -\tfrac{1}{2}(x^2+y^2)\,\sigma+gtT_6+\tfrac{1}{2}gt^2T_4.
\end{aligned}$$

Each of these conserved densities has a simple physical meaning, to be made clearer by the integral identities that follow. In particular, T_3 relates to energy conservation, T_4 and T_5 to the conservation of horizontal and vertical impulse, about which more will be said in §5. T_6 and T_7 connect kinematic properties of the displaced mass with the components of impulse, and T_8 relates to conservation of angular impulse. Given any T_j, the complete conservation law can easily be found from (9). For example, the fourth conservation law is

$$\left.\begin{aligned}
&(y\sigma)_t+\xi_x+\eta_y = 0\\
\text{with}\qquad &\xi = uy\sigma+\tfrac{1}{2}\rho(u^2-v^2)+\tfrac{1}{2}y\rho_y(u^2+v^2)+gy(\rho-\rho_0),\\
&\eta = vy\sigma+\rho uv-\tfrac{1}{2}y\rho_x(u^2+v^2),
\end{aligned}\right\} \tag{25}$$

where, as before, u and v should be understood as functional transformations of ω by virtue of (2). This result will be discussed further in §5.

The remaining conservation laws comparable with (25) need not be spelled out: the mere fact of their existence will suffice for the record. To complete the present account, however, integral forms of the eight conservation laws will be noted. Adding to the integrals I_1 and I_2 introduced in §1, we consider

$$\begin{aligned}
I_3 &= \int_D H\,\mathrm{d}x\,\mathrm{d}y = \hat{H} \quad \text{(total energy)},\\
I_4 &= \int_D y\sigma\,\mathrm{d}x\,\mathrm{d}y \quad \text{(horizontal impulse)},\\
I_5 &= \int_D (-x\sigma)\,\mathrm{d}x\,\mathrm{d}y \quad \text{(vertical impulse)},\\
I_6 &= \int_D x(\rho-\rho_0)\,\mathrm{d}x\,\mathrm{d}y \quad (\bar{x}I_1),\\
I_7 &= \int_D y(\rho-\rho_0)\,\mathrm{d}x\,\mathrm{d}y \quad (\bar{y}I_1),\\
I_8 &= \int_D \{-\tfrac{1}{2}(x^2+y^2)\,\sigma\}\,\mathrm{d}x\,\mathrm{d}y \quad \text{(moment of impulse)}.
\end{aligned}$$

The following theorem will refer to localized motions for which each integral converges although the measure of D is infinite. Note that the last four are integrals of the respective parts of T_5–T_8 not explicitly dependent on t. Note also that $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid of the displaced mass, and that gI_7 represents the total potential energy relative to the quiescent state of the system. Finally note that I_1 need not be zero since we allow the possibility of ρ having a measure distribution different from ρ_0, to which ρ converges at large distances; thus the motion may be started by the addition of fluid, not merely by stirring up the static equilibrium represented by ρ_0.

Taking D to be the infinite strip $\mathbb{R} \times (0, h)$ bounded by rigid horizontal planes, we further need to define the boundary integrals

$$B_5 = \int_{\mathbb{R}} [\tfrac{1}{2}\rho u^2]_0^h \, \mathrm{d}x, \qquad B_6 = \int_{\mathbb{R}} [\rho u y]_0^h \, \mathrm{d}x,$$

$$B_7 = -\int_{\mathbb{R}} [\rho u]_0^h \, x \, \mathrm{d}x, \quad B_8 = -\int_{\mathbb{R}} [\tfrac{1}{2}\rho u^2]_0^h \, x \, \mathrm{d}x,$$

where $[\,.\,]_0^h$ denotes the difference between evaluations at $y = h$ and $y = 0$. From the definition (1) of σ it follows that

$$I_4 + B_6 = \int_D \rho u \, \mathrm{d}x \, \mathrm{d}y, \tag{26}$$

which expresses the total horizontal momentum of the fluid in D; and similarly

$$I_5 + B_7 = \int_D \rho v \, \mathrm{d}x \, \mathrm{d}y. \tag{27}$$

Theorem 5. For any localized free motion of a heterogeneous, incompressible and inviscid fluid between horizontal rigid planes, the eight quantities I_j satisfy

$$I_1 = \text{const.}, \qquad I_2 = \text{const.},$$

$$I_3 = \text{const.}, \qquad I_4 = \text{const.},$$

$$\frac{\mathrm{d}I_5}{\mathrm{d}t} = -gI_1 + B_5, \quad \frac{\mathrm{d}I_6}{\mathrm{d}t} = I_4 + B_6,$$

$$\frac{dI_7}{\mathrm{d}t} = I_5 + B_7, \qquad \frac{\mathrm{d}I_8}{\mathrm{d}t} = -gI_6 + B_8.$$

A detailed proof of this theorem will be omitted, being very straightforward. Upon differentiation of each integral I_j with respect to t and substitution from (9) for ρ_t and σ_t (or more conveniently for $\mathrm{D}\rho/\mathrm{D}t$ and $\mathrm{D}\sigma/\mathrm{D}t$), integrations by parts lead to integrated terms that, except for those specified in the theorem, make no contribution because $v = \rho_x = 0$ on the solid boundaries and because various densities such as $|x(\rho - \rho_0)|$ and $|x(u, v)|$ vanish as $|x| \to \infty$ in the case of a localized motion. The complete conservation laws corresponding to the T_j of Theorem 4 are in effect exposed by these calculations.

A version of Theorem 5 applying to x-periodic motions is demonstrable just as readily. With all integrals redefined over one period the statement of the Theorem is unchanged. Theorem 5 is comparable with Theorem 6.2 in Benjamin & Olver (1982), which applies to localized water-wave motions in an infinite ocean lying on a horizontal rigid bottom at finite depth; but there the numbering of the I_j is different.

The present I_2 has no distinct counterpart there, and the I_7 there is a quantity labelled viral that is not matched by the Boussinesq system.

The results in Theorem 5 can be extended to the case that D is the whole of $\mathbb{R}^2$, and the boundary integrals B_5 and B_8 are in fact replaceable by zero in the statement of the theorem. The meaning of the last five conservative properties then requires careful interpretation, however, which aspect will be discussed in §5. It should be noted finally that energy conservation as expressed by $\mathrm{d}\hat{H}/\mathrm{d}t = 0$ holds if D has finite rigid boundaries of whatever form, provided only that g and therefore H are independent of t explicitly. But total energy is not conserved when g measured in the frame (x, y) varies with t, as when the fluid fills a rigid container that is vibrated vertically.

5. Impulse

In our discussion of the connections between Hamiltonian structure, symmetries and conservation laws, the density $T_4 = y\sigma$ has been exhibited as the natural representative of horizontal impulse for the Boussinesq model. Let us re-introduce the notation m_1 for $y\sigma$ as used in the previous account (B, p. 35). Also, referring to Theorem 3 coupled with the fourth entries in Theorems 1 and 4, let us recall its salient property to be

$$\mathbf{J}\mathscr{E}m_1 = -\boldsymbol{\omega}_x. \tag{28}$$

Correspondingly, we recall that

$$\mathbf{J}\mathscr{E}m_2 = -\boldsymbol{\omega}_y, \tag{29}$$

where $m_2 = -x\sigma$ is the part of T_5 that does not depend explicitly on t. Note that m_2 is not by itself a conserved density, but in relation to Hamiltonian structure and the symmetry group G_5 it appears to be the natural representative of vertical impulse. The physical interpretations of m_1 and m_2 will now be examined, together with that of $-\frac{1}{2}(x^2+y^2)\,\sigma = N$, say, which is the part of T_8 not explicitly dependent on t and will be identified with the impulsive couple for the motion. The following account extends Kelvin's original conceptions about impulse, as expounded by Lamb (1932, §152), but care is needed to adapt them to the Boussinesq model. Distinct considerations apply to the three cases where D is the whole of $\mathbb{R}^2$, where D is the upper or lower half-space and where D is an infinite strip between rigid horizontal planes.

In the first two cases the delicacy of the issues can be appreciated from a discussion by Benjamin & Olver (1982, §6.5), who dealt with the components of impulse for localized water-wave motions in the case of infinite depth. It was shown that plain meanings can then be put on all the conservation laws for water waves only if the far-field of the velocity potential has no dipole component oriented in the horizontal direction. This necessary condition is physically reasonable because, if it were not satisfied, the wave motion would have infinite angular momentum and so could not be generated by finite forces acting on any bounded stretch of the free surface. By means of a simple example, however (Benjamin & Olver 1982, footnote to p. 175), it was shown how infinite angular velocity is an admissible feature of motions in a half-space provided the boundary is rigid: the reaction of an impulsive pressure field generating the motion from rest can then impart an infinite impulsive couple.

5.1. *Asymptotic specifications*

To translate this interpretation to the present model, we need to be specific about the density distribution at infinity. When D is the upper half-space the most realistic specification is that ρ tends to a minimum value $\rho_1 > 0$ asymptotically as $y \to \infty$; and when D is the lower half-space ρ is asymptotic to a maximum ρ_2 as $y \to -\infty$. These specifications are to be combined in the case that D is the whole of $\mathbb{R}^2$. Then in the limit $r = (x^2+y^2)^{\frac{1}{2}} \to \infty$ taken along any ray with $\theta = \arctan(y/x)$ other than 0 or π, stratification is disengaged but $\lim_{r\to\infty} \rho$ is ρ_1 or ρ_2 accordingly as $0 < \theta < \pi$ or $-\pi < \theta < 0$.

Now, for any localized motion of an incompressible stratified fluid filling $\mathbb{R}^2$ or a half-space, there can be no monopole velocity field because it would entail infinite kinetic energy (i.e. $\psi \sim C \ln(1/r)$ as $r \to \infty$ is inadmissible). For $D = \mathbb{R}^2$, therefore, the asymptotic form of the stream function $\psi = \mathrm{B}_{(\rho)}\sigma$ for $r \to \infty$ must be

$$\left.\begin{aligned} \psi &\sim \frac{(a_1 y + b_1 x)}{r^2} \quad \text{if} \quad 0 < \theta < \pi \\ &\sim \frac{(a_2 y + b_2 x)}{r^2} \quad \text{if} \quad -\pi < \theta < 0. \end{aligned}\right\} \tag{30}$$

Note that a_1 and a_2 are the coefficients of horizontally directed dipoles, b_1 and b_2 those of vertically directed ones. Considering the density of (anticlockwise) angular momentum and recalling that $(u, v) = (\psi_y, -\psi_x)$ we deduce from (30) that

$$\rho(xv - yu) \sim \rho_1 \psi \tag{31}$$

in the upper half-space and the same with ρ_2 in the lower. This asymptotic estimate shows plainly that the integrals of $\rho(xv-yu)$ over the upper and lower half-spaces respectively diverge unless $a_1 = 0$ and $a_2 = 0$. But no contribution to total angular momentum arises from the terms in b_1 and b_2.

In the case that D is the upper (respectively lower) half-space, allowance must nevertheless be made for the possibility that $a_1 \neq 0$ (respectively $a_2 \neq 0$) and consequently the motion has limitless angular momentum. In this case, for example, solitary waves are known to be possible, having the far-fields of horizontal dipoles (cf. Benjamin 1967, p. 579). Although perhaps surprising at first, this fact about angular momentum can be explained quite satisfactorily in the way already noted: it is artificial, of course, but is an intrinsic attribute of mathematical models where an incompressible fluid fills a rigidly bounded, two-dimensional half-space.

5.2. *The case $D = \mathbb{R}^2$*

In this case, on the other hand, infinite angular momentum is unacceptable for localized wave motions in the Boussinesq model.† Accordingly, in order to ensure cancellation of the infinite positive and negative angular momenta in the upper and lower far-fields, (30) and (31) show that the condition

$$\rho_1 a_1 = \rho_2 a_2 \tag{32}$$

must be satisfied in every realistic example – that is for every wave motion generated by finite forces.

Another, more obvious special property when $D = \mathbb{R}^2$ is implied by (30), that is

† It deserves emphasis that an unbounded fluid under gravity is an abstract model, approximating practical situations where the bottom of the fluid is far below its density-stratified region. To expose the properties most relevant to such cases, the mathematical arguments that follow need to allow for limits taken as $y \to -\infty$ in the model; they proceed abstractly irrespective of and without controverting the practical necessity of a bottom somewhere. The classic model for waves on deep water has just the same character.

by the assumption of no monopole far field as is necessary for finite kinetic energy. From the definition (1) of σ it follows that

$$\int_{D'} \sigma \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial D'} \rho(u\,\mathrm{d}x + v\,\mathrm{d}y) \tag{33}$$

for every bounded $D' \subset \mathbb{R}^2$ with boundary $\partial D'$. This contour integral can be interpreted as a density-weighted circulation around $\partial D'$. Taking D' to be a large disk centred on the origin, one sees from (30) that for any a_1 and a_2 the integral is $O(1/r)$ as $r \to \infty$; moreover, the condition (32) implies it to be $o(1/r)$ as $r \to \infty$. Thus, the limit of (33) shows the assumption of finite energy to require

$$I_2 = \int_{\mathbb{R}^2} \sigma \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{34}$$

Returning momentarily to the case where D is a half-space, however, note that I_2 need not then be zero since circulation can be accumulated along the boundary $y = 0$.†

Generalizing the account by Lamb (1932, §152) let us now represent a free motion by the fictitious field of impulsive force $(X, Y)(x, y, t)$ *per unit volume* that would generate it instantaneously from rest. More precisely, at any time $t = t_0+$, say, the motion (u, v) is appreciated to be as if $|(u, v)| = 0$ at $t = t_0-$ and there were applied an external force field $(X, Y)(x, y, t_0)\,\delta(t - t_0)$, where δ is the Dirac distribution. The density ρ of the incompressible fluid is not, of course, affected by this process, and an impulsive pressure $p'(x, y, t_0)\,\delta(t-t_0)$ will arise from it. The state of motion thus given at $t = t_0+$ requires

$$\rho u = X - \frac{\partial p'}{\partial x}, \quad \rho v = Y - \frac{\partial p'}{\partial y}, \tag{35}$$

and so

$$\sigma = -X_y + Y_x. \tag{36}$$

Further, arbitrary specifications would be required to make (X, Y) and p' fully determinate from (35); but for what follows there is no need to calculate them and (35) is a sufficient description.

The present problem does not allow us in general to bound $\operatorname{supp}\sigma$. So, in contrast with the classic problem of finite vortex systems in an infinite fluid (Lamb, §152), it cannot be supposed that X and Y are zero outside some bounded region. However, a satisfactorily wide interpretation can be based on the assumption that $|(X, Y)| = O(1/r^{3+\epsilon})$ with $\epsilon > 0$ as $R \to \infty$ or, somewhat more generally, that

$$xY - yX = O(1/r^{2+\epsilon}) \quad \text{as } r \to \infty. \tag{37}$$

This assumption is consistent with (36) and (34) for $D = \mathbb{R}^2$; and for this case also (36) and (37) lead to

$$\hat{m}_1 = \int_{\mathbb{R}^2} y\sigma \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} X \,\mathrm{d}x\,\mathrm{d}y, \tag{38}$$

$$\hat{m}_2 = \int_{\mathbb{R}^2} (-x\sigma) \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} Y \,\mathrm{d}x\,\mathrm{d}y, \tag{39}$$

$$\hat{N} = \int_{\mathbb{R}^2} \{-\tfrac{1}{2}(x^2+y^2)\,\sigma\} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} (xY - yX) \,\mathrm{d}x\,\mathrm{d}y. \tag{40}$$

† It is at first sight tempting to explain this case consistently with (34) by considering a sign-reversed image of σ in the rigid boundary. This notion is well known to apply to vorticity in homogeneous fluids. But in the present model stratification invalidates such a rationale.

To confirm (38) the identity $y\sigma = X-(yX)_y+(yY)_x$ is integrated over a disk of radius r, whence the divergence on the right gives a contour integral that vanishes in the limit $r\to\infty$ by virtue of (37). The other two results are proved in the same way.

Thus, for $D = \mathbb{R}^2$, the quantity $\hat{m}_1$ ($\equiv I_4$) is shown to be the total horizontal impulse imparted to the system by the forces generating the motion. Likewise $\hat{m}_2$ ($\equiv I_5$) is the total vertical impulse; and $\hat{N}$ ($\equiv I_8$) is the total impulsive couple. In the present case $\hat{N}$ must also equal total angular momentum, whose finiteness is ensured by the condition (32). A further interpretation of the integrals on the right of (38) and (39) is available from (35), which shows that

$$\int_{\mathbb{R}^2} X\,\mathrm{d}x\,\mathrm{d}y = \lim_{r\to\infty}\left\{\int_{D'}\rho u\,\mathrm{d}x\,\mathrm{d}y+\oint_{\partial D'}p'\,\mathrm{d}y\right\}, \tag{41}$$

$$\int_{\mathbb{R}^2} Y\,\mathrm{d}x\,\mathrm{d}y = \lim_{r\to\infty}\left\{\int_{D'}\rho v\,\mathrm{d}x\,\mathrm{d}y-\oint_{\partial D'}p'\,\mathrm{d}x\right\}, \tag{42}$$

were D' is a disk of radius r. Each of the four integrals on the right is generally indeterminate as $r\to\infty$, but the given combinations of them must converge to the limits $\hat{m}_1$ and $\hat{m}_2$. In other words, as is well known in the case of homogeneous incompressible fluids, impulse can be interpreted – albeit somewhat vaguely – as the difference between the total momentum and the reaction of the impulsive pressure at infinity. Confirming a simple fact already mentioned, the corresponding reduction of the integral on the right of (40) shows $\hat{N}$ to equal the integral of $yv-xu$ over $\mathbb{R}^2$; the related contour integral involving p' is automatically zero when $\partial D'$ is a circle.

The laws of net impulse conservation included in Theorem 5 can now be extended to the case $D = \mathbb{R}^2$. In particular, from the explicit local conservation law (25), an integration and a reference to (30) confirms I_4 ($\equiv \hat{m}_1$) to be a constant of any free motion in $\mathbb{R}^2$. It is also easily found in the present case that counterparts of the boundary integrals B_5 and B_8 in Theorem 5 are zero, so that we have

$$\frac{\mathrm{d}\hat{m}_2}{\mathrm{d}t} = -gI_1 \tag{43}$$

and

$$\frac{\mathrm{d}\hat{N}}{\mathrm{d}t} = -gI_6. \tag{44}$$

On the other hand the equations for I_6 and I_7 are more difficult to adapt and interpret, needing special consideration as follows. The corresponding pair of conservation laws for water waves also demands careful appraisal (see Benjamin & Olver 1982, §6.5).

In the first place, referring to any bounded $D'\subset\mathbb{R}^2$, consider the purely kinematic identities

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{D'}x(\rho-\rho_0)\,\mathrm{d}x\,\mathrm{d}y &= -\int_{D'}x(u\rho_x+v\rho_y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{D'}\rho u\,\mathrm{d}x\,\mathrm{d}y+\oint_{\partial D'}x\rho(v\,\mathrm{d}x-u\,\mathrm{d}y).\end{aligned} \tag{45}$$

In the absence of additional, dynamically irrelevant specifications about conditions at infinity, the first integral on the right does not remain determinate as D' is expanded to fill $\mathbb{R}^2$; and the contour integral is not reducible to the one in (41). Writing $\rho u = y\sigma-(y\rho v)_x+(y\rho u)_y$ in the first integral, however, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D'}x(\rho-\rho_0)\,\mathrm{d}x\,\mathrm{d}y = \int_{D'}y\sigma\,\mathrm{d}x\,\mathrm{d}y+\oint_{\partial D'}\rho\{(xv-yu)\,\mathrm{d}x-(xu+yv)\,\mathrm{d}y\}, \tag{46}$$

where the first integral on the right remains determinate as $D' \to \mathbb{R}^2$ and its limit $\hat{m}_1$ is known to be a constant of the motion. The contour integral can be evaluated asymptotically by use of (30), and thus the result from (46) in the limit as $D' \to \mathbb{R}^2$ is found to be

$$\frac{\mathrm{d}I_6}{\mathrm{d}t} = \hat{m}_1 - \pi(\rho_1 a_1 + \rho_2 a_2). \tag{47}$$

Here I_6 denotes the integral of $x(\rho - \rho_0)$ over $\mathbb{R}^2$ and the final terms on the right are reducible by (32) to $-2\pi\rho_1 a_1$.

In the same way the result corresponding to the seventh in Theorem 5 is found to be

$$\frac{\mathrm{d}I_7}{\mathrm{d}t} = \hat{m}_2 + \pi(\rho_1 b_1 + \rho_2 b_2) \tag{48}$$

when $D = \mathbb{R}^2$. Here $\hat{m}_2$ is not in general constant but varies according to (43).

A simple check on these results is provided by the case of a homogeneous fluid, say with $\rho = 1$. Then I_6 and I_7 are of course null. But σ then reduces to vorticity $\zeta = -\Delta\psi$, whence an application of Green's theorem shows that the terms on the right of (47) cancel (cf. (53) below), as also do those on the right of (48). The results are otherwise strange. Although the first term on the right of (47) is constant, the term attributable to the dipole far field cannot in general be inferred to remain constant. It is in fact only in the case of a homogeneous fluid that, with $I_2 = 0$, the strength of the far field is determined uniquely by the moments $\hat{m}_1$ and $\hat{m}_2$ of the σ-distribution. Thus, contrary to what might be expected from offhand comparison with the water-wave problem (cf. Benjamin & Olver 1982, eqn. (6.24)), the conservation law (47) does *not* imply that $\mathrm{d}I_6/\mathrm{d}t = \text{const.}$†

This conclusion may be reinforced by examining the consequences of more precise conditions at infinity, as was done in Benjamin & Olver's discussion of water waves (1982, §6.5). For instance, if $\partial D'$ is taken to be a *rigid* circular boundary whose radius $r \to \infty$, the contour integral on the right of (45) is zero. But the requisite modification of ψ as $r \to \infty$ (Benjamin & Olver 1982, p. 174) is found to alter the other part of the contour integral in (46) so that the result (47) is regained. The final outcome is also the same if the boundary at infinity is taken to be compliant in such a way that hydrostatic pressure is exactly maintained upon it (Benjamin & Olver 1982, p. 172).

5.3. *The case of a half-space*

As already noted, this case is in several respect more curious than the last. In general a_1, or respectively a_2, is not zero, so that angular momentum is unbounded, and there are non-zero counterparts of all the boundary terms B_5–B_8 in Theorem 5. The four integrals I_1–I_4 are nevertheless easily confirmed to be constants of any free motion in the upper or lower half-space with rigid boundary $y = 0$; and $I_4 \equiv \hat{m}_1$ representing total horizontal impulse is again identifiable as in (38) with the integral of X over D. Furthermore, expressions corresponding to (41) and (42) obviously hold for a half-space. For horizontal impulse, therefore, the same two-fold interpretation as before remains applicable.

† It is noteworthy that this feature of the conservation law associated with horizontal Galilean invariance is not reproduced by the model commonly known as the Benjamin–Ono equation, which is an approximation for weakly nonlinear and dispersive long waves in stratified-fluid systems of the type treated exactly here (B, Section 3). Localized solutions u of the equation satisfy the simpler conservation law $(\mathrm{d}/\mathrm{d}t)\int_{\mathbb{R}} ux\,\mathrm{d}x = \int_{\mathbb{R}} \tfrac{1}{2}u^2\,\mathrm{d}u = \text{const.}$ This lack of formal correspondence is not surprising because the B–O equation is known to have many conservative properties not shared by the exact physical systems that it simulates.

As regards $I_5 \equiv \hat{m}_2$ the interpretation is less plain. Writing for the upper half-space $S+ = R \times [0, \infty)$ and for the lower $S- = R \times (-\infty, 0]$, we find from (36), (37) and then the respective version of (42) that

$$\hat{m}_2 = \int_{S\pm} (-x\sigma)\,\mathrm{d}x\,\mathrm{d}y = \int_{S\pm} Y\,\mathrm{d}x\,\mathrm{d}y \mp \int_R xX_{y=0}\,\mathrm{d}x$$
$$= \int_{S\pm} \rho v\,\mathrm{d}x\,\mathrm{d}y + \oint_{\partial S\pm} p'\,\mathrm{d}x \mp \int_R xX_{y=0}\,\mathrm{d}x. \tag{49}$$

Note that for the contour integral $\partial S+$ includes the x-axis taken positively, and $\partial S-$ the x-axis taken negatively. Just as

$$I_2 = \int_{S\pm} \sigma\,\mathrm{d}x\,\mathrm{d}y = \pm \int_R X_{y=0}\,\mathrm{d}x = \pm \int_R (\rho u)_{y=0}\,\mathrm{d}x$$

is generally not zero, neither is the last integral on the right of (49), and this component of $\hat{m}_2$ has no simple physical meaning. Precisely in keeping with the fifth item in Theorem 5, however, the conservation law for net vertical impulse in a half-space is found from (9) coupled with (30) to be

$$\frac{\mathrm{d}\hat{m}_2}{\mathrm{d}t} = -gI_1 \mp \int_R \tfrac{1}{2}(\rho u^2)_{y=0}\,\mathrm{d}x. \tag{50}$$

In the present case the next two conservation laws are clarified as follows by another expression for the dipole far field; and as may be expected it will be confirmed that b_1, or respectively b_2, is zero owing to the presence of the rigid horizontal boundary. Let us recall from (1) that vorticity ζ is related to the dependent variables ρ and σ by

$$\zeta = -\Delta\psi = \rho^{-1}(\sigma + \nabla\rho\cdot\nabla\psi), \tag{51}$$

where $\psi = \mathrm{B}_{(\rho)}\sigma$. Let us also recall that when D is either $S+$ or $S-$, and the boundary conditions $\psi = 0$ on $y = 0$ and $\psi \to 0$, $|\nabla\psi| \to 0$ as $r \to \infty$ are incorporated, the integral operation $(-\Delta)^{-1}: \mathrm{L}^2(D) \to C(D)$ has the Green function

$$k(x, y; \tilde{x}, \tilde{y}) = \frac{1}{4\pi} \ln\left\{\frac{(x-\tilde{x})^2 + (y+\tilde{y})^2}{(x-\tilde{x})^2 + (y-\tilde{y})^2}\right\}.$$

As $r = (x^2+y^2)^{\frac{1}{2}} \to \infty$ for fixed $(\tilde{x}, \tilde{y})$, we hence have

$$k(x, y; \tilde{x}, \tilde{y}) = \frac{y\tilde{y}}{\pi r^2} + O\left(\frac{1}{r^2}\right), \tag{52}$$

which establishes two facts of interest. First, on the (already implicit) assumption that $\zeta = O(r^{-3-\epsilon})$ with $\epsilon > 0$, it plainly follows that when $D = S+$ the dipole coefficient a_1 in (30) is given by

$$a_1 = \frac{1}{\pi} \int_{S+} y\zeta\,\mathrm{d}x\,\mathrm{d}y, \tag{53}$$

in which the expression (51) for ζ may be substituted. A corresponding formula for a_2 holds when $D = S-$. In the light of (51) these simple formulae highlight that a_1, or respectively a_2, varies with t in nearly all cases.† The second, more helpful fact made conspicuous by (52) is that b_1, or respectively b_2, is always zero when D is a half-space.

† An exception is when σ is an odd function and ρ an even function of x at $t = 0$. Reference to (9) shows ω to stay in this classification for $t > 0$, so that a_1 or a_2 remains zero.

In the same way as (47) was derived for $\mathbb{R}^2$, the sixth conservation law for $S+$ is found to be

$$\frac{\mathrm{d}I_6}{\mathrm{d}t} = \hat{m}_1 - \pi a_1, \tag{54}$$

and for $S-$ the only change in the statement of the law is that a_2 replaces a_1. Here $\hat{m}_1$ defined as an integral over $S+$ or $S-$ is constant, but a_1 or a_2 is generally not.

Next, in consequence of the vertical dipole strength at infinity being zero for a half-space, the seventh conservation law corresponding to (48) is found to be

$$\frac{\mathrm{d}I_7}{\mathrm{d}t} = \hat{m}_2 \pm \int_{\mathbb{R}} (\rho u)_{y=0}\, x\,\mathrm{d}x. \tag{55}$$

The integral over $\mathbb{R}$ is not absolutely convergent since $u(x,0,t) \sim a_1/x^2$, or respectively a_2/x^2, as $x \to \pm\infty$; but this integral can be confirmed to converge conditionally by virtue of cancellations in the component integrals over $(-R,0]$ and $(0,R)$ as $R\to\infty$. On this understanding about conditional convergence, (55) may be combined with (50) to give

$$\frac{\mathrm{d}^2 I_7}{\mathrm{d}t^2} = -gI_1 \pm \int_{\mathbb{R}} (\rho u_t\, x - \tfrac{1}{2}\rho u^2)_{y=0}\,\mathrm{d}x.$$

A simple interpretation of this result follows upon recognition from the original Euler equations that on $y=0$ we have $(\rho u)_t = -(\tfrac{1}{2}\rho u^2 + p_0^*)_x$, where p_0^* is the pressure on the boundary relative to hydrostatic pressure. Hence, with use of the fact that

$$\lim_{x\to\infty} x p_0^*(x,t) = \rho_0(0)\,\frac{\mathrm{d}a_1}{\mathrm{d}t}$$

is the same as the limit for $x\to-\infty$ (similarly with a_2 when $D = S-$), an integration by parts leads to

$$\frac{\mathrm{d}^2 I_7}{\mathrm{d}t^2} = -gI_1 \pm \int_{\mathbb{R}} p_0^*\,\mathrm{d}x. \tag{56}$$

Notwithstanding that the integral of p_0^* is only conditionally convergent, equation (56) makes good sense physically (cf. Benjamin & Olver 1982, eqn. (6.14)). If the excess mass I_1 is non-zero, the left side of (56) can be understood as $I_1\,\mathrm{d}^2\bar{y}/\mathrm{d}t^2$. Thus excess mass times the vertical acceleration of its centroid is shown to equal a net vertical force upwards, namely the total reaction of the boundary to the dynamic pressure upon it less the excess weight of the fluid.

For a half-space, unlike $\mathbb{R}^2$ or an infinite strip, it becomes a *sine qua non* to present the eighth and final conservation law in terms of $\hat{N} \equiv I_8$ rather than total angular momentum, which is typically unbounded. Note first from (36) and (40) that

$$\begin{aligned}\hat{N} &= \int_{S\pm} \tfrac{1}{2}(x^2+y^2)\,(X_y - Y_x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{S\pm} (xY - yX)\,\mathrm{d}x\,\mathrm{d}y \mp \int_{\mathbb{R}} \tfrac{1}{2}x^2 X_{y=0}\,\mathrm{d}x.\end{aligned}$$

Here the first integral on the right has an obvious physical interpretation, but the second does not and is incapable of reduction to anything simpler. (The substitution of $X_{y=0} = (\rho u + p')_{y=0}$ in the integrand tells us nothing because the integrals so obtained are individually meaningless.) From the second row of (9), however, it is easily shown that

$$\frac{\mathrm{d}\hat{N}}{\mathrm{d}t} = -gI_6 \mp \int_R \tfrac{1}{2}x(\rho u^2)_{y=0}\,\mathrm{d}x. \tag{57}$$

This result might be expected from the final item in Theorem 5; and since $u \sim a_1/x^2$ or a_2/x^2 as $|x| \to \infty$ on $y = 0$, the integral over $\mathbb{R}$ is absolutely convergent. Note finally that the conservation law (57) cannot be translated into terms of the moment of dynamic pressure on the boundary, as might at first sight seem feasible (see (60) below). In fact the integral of xp_0^* over $\mathbb{R}$ typically does not exist.

5.4. *Motions between horizontal boundaries*

The integral conservation laws for this case have been listed in Theorem 5, but three points of interpretation deserve attention regarding impulse. Note first that whereas total horizontal impulse $I_4 \equiv \hat{m}_1$ is a constant of any free motion, total horizontal momentum as given by (26) is *not*, even though in the present case it remains always determinate for localized motions.† This curious fact about the Boussinesq model accords with our conclusion that $\mathrm{d}I_6/\mathrm{d}t$ is not constant for typical motions in $\mathbb{R}^2$ or a half-space, and it is certainly not in conflict with general dynamical principles – as might perhaps appear at first sight. It can be explained as follows.

Differentiating the boundary integral B_6 in the identity (26) and using the x-component of the Euler equations, one obtains

$$\frac{\mathrm{d}B_6}{\mathrm{d}t} = h\int_R \partial_t(\rho u)_{y=0}\,\mathrm{d}x = -h\int_R \partial_x(\tfrac{1}{2}\rho u^2 + p^*)_{y=0}\,\mathrm{d}x = h(p^*_{-\infty} - p^*_{\infty}). \tag{58}$$

Here p^*_{∞} and $p^*_{-\infty}$ are the pressure levels as $x \to \infty$ and $x \to -\infty$ relative to hydrostatic pressure in the quiescent state of the whole system; and while only their difference can have any dynamic significance there is no reason in general for it to be zero or take any other constant value. Horizontal impulse is invariant according to the fourth item of Theorem 5, which in view of (26) implies the invariance of horizontal momentum *minus* the integral of the force (58) with respect to time. (This property also follows immediately from an integration of the Euler equation over the infinite strip.) But the two quantities whose difference thus recovers I_4 are not required separately to be invariant. The sixth item of Theorem 5 shows that $\mathrm{d}I_6/\mathrm{d}t$ equals horizontal momentum and so is generally not constant for a heterogeneous fluid. The case of a homogeneous fluid is exceptional, however, because I_6 and horizontal momentum are then null quantities; hence, because I_4 is still invariant in this case, (58) implies that $p^*_{\infty} = p^*_{-\infty}$ permanently.

To conclude let us note alternative forms of the seventh and eighth conservation laws with boundary integrals expressed in terms of the dynamic pressure p^*. It should again be appreciated that, except for an arbitrary additional function of t alone which has no effect on the present results, p^* is determined by the Euler equations and thus can be considered as a functional transformation of the Hamiltonian variables ρ and σ which determine ψ. Combining the seventh and fifth items in Theorem 5 and using

† For any localized motion of a *homogeneous* incompressible fluid between parallel planes, total momentum must be zero by virtue of the condition $u_x + v_y = 0$. But it evidently can be non-zero for a heterogeneous incompressible fluid.

the Euler equations to reduce the boundary terms, or proceeding directly from the latter, one obtains

$$\frac{\mathrm{d}^2 I_7}{\mathrm{d}t^2} = -gI_1 - \int_R [p^*]_0^h \,\mathrm{d}x. \tag{59}$$

The integral of the difference in dynamic pressure between top and bottom can be presumed to converge, although of course the integral of either pressure typically does not exist. The result (59) has an obvious physical interpretation as exemplified after (56).

The form of the final conservation law given in Theorem 5 is the simplest in relation to Hamiltonian structure, but for this law too an equivalent form exposing its physical meaning is found upon the introduction of p^* by means of the Euler equations. Writing $N = xv - yu - (\frac{1}{2}r^2\rho v)_x + (\frac{1}{2}r^2\rho u)_y$ in the given form, integrating by parts and then using the Euler equations, or proceeding directly from the latter, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R\int_0^h (xv - yu)\,\mathrm{d}x\,\mathrm{d}y = -gI_6 - \int_R x[p^*]_0^h \,\mathrm{d}x. \tag{60}$$

As can be expected, the rate of change of total (anticlockwise) angular momentum is thus shown to equal the net couple composed of the excess weight of the fluid and the difference of dynamic pressures on the boundaries.

6. Steady waves

The simplest group-invariant solutions of (9) are periodic and solitary waves of permanent form travelling at a constant velocity c in the x-direction, so that $\boldsymbol{\omega} = \boldsymbol{\omega}(x - ct, y)$. The variational characterization of such motions was reviewed abstractly in B (§4.2) for the case where, as in (9), the Hamiltonian operator $\mathbf{J}$ is $\boldsymbol{\omega}$-dependent. Several details of the present example deserve to be noted, however, particularly since an interesting practical application is in prospect.

Because $\boldsymbol{\omega}_t = -c\boldsymbol{\omega}_x$ for steady waves, a combination of (9) and (28) shows them to be solutions of

$$\mathbf{J}\mathscr{E}(H - cm_1) = 0, \tag{61}$$

which is the Euler–Lagrange equation for a variational principle of special type. Recall that $\mathscr{E}H$ is given by (8) and $\mathscr{E}m_1 = [0, y]^{\mathrm{T}}$. Here, attention will be concentrated on the principle itself, but it should first be acknowledged that (61) is a concise representation of steady-wave equations that are otherwise well known. The first row of (61) is $\partial(\rho, \psi - cy) = 0$ and so implies ρ to be a function of $\Psi = \psi - cy$ alone, which variable is the stream function in a frame of reference moving with the wave. The second row regains the equation for Ψ discovered by Dubreil-Jacotin (1935), as studied by Long (1953), Yih (1965, chapter 3) and many others. For details of the latter implication reference may be made to B (p. 36).

To appreciate the variational principle, observe first that (61) is a pair of scalar equations for the Hamiltonian variables ρ and σ, not for ψ which is still reckoned as the transformation $\mathrm{B}_{(\rho)}\sigma$ of these variables. Note also that t-dependence can be left implicit in (61): a solution found as $\boldsymbol{\omega}(x, y)$ provides a solution $\boldsymbol{\omega}(x - ct, y)$ of (9). Accordingly, relative to a particular solution $\bar{\boldsymbol{\omega}} = [\bar{\rho}, \bar{\sigma}]^{\mathrm{T}}$ of (61), consider the class (semigroup) $\mathscr{C}$ of varied functions $\boldsymbol{\omega}(x, y; s)$ defined as follows. With parameter $s \geqslant 0$ these functions are solutions of the linear Cauchy problems

$$\boldsymbol{\omega}_s = -\mathbf{J}(\boldsymbol{\omega})\boldsymbol{\Lambda}, \quad \boldsymbol{\omega}(0) = \bar{\omega},$$

in which $\boldsymbol{\Lambda} = [\lambda, \mu]^{\mathrm{T}}$ is a continuously differentiable, bounded but otherwise arbitrary function of x, y and s.† In components these infinitely various problems are shown by (10) to be

$$\left.\begin{aligned} \rho_s &= \partial(\rho, \mu), & \rho(0) &= \bar{\rho}, \\ \sigma_s &= \partial(\rho, \lambda) + \partial(\sigma, \mu), & \sigma(0) - &= \bar{\sigma}, \end{aligned}\right\} \tag{62}$$

and solutions will be supposed representable by successive approximations in powers of s. Thus we consider $\rho(s) = \bar{\rho} + s\dot{\rho} + \frac{1}{2}s^2\ddot{\rho} + \ldots$ and similarly for $\sigma(s)$, recognizing the first (infinitesimal) variation of $\boldsymbol{\omega}$ to have components

$$\left.\begin{aligned} \dot{\rho} &= \left[\frac{\partial\rho(s)}{\partial s}\right]_{s=0} = \partial(\bar{\rho}, \bar{\mu}), \\ \dot{\sigma} &= \left[\frac{\partial\sigma(s)}{\partial s}\right]_{s=0} = \partial(\bar{\rho}, \bar{\lambda}) + \partial(\bar{\sigma}, \bar{\mu}). \end{aligned}\right\} \tag{63}$$

Here $\bar{\lambda} = \lambda(0)$ and $\bar{\mu} = \theta(0)$ denote arbitrary functions of x, y alone.

Now consider the functional $\hat{H} - c\hat{m}_1$ evaluated in this class of variations. We have for all $s \geqslant 0$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\hat{H} - c\hat{m}_1) = \int_D \{\mathscr{E}_\rho(H - cm_1)\rho_s + \mathscr{E}_\sigma(H - cm_1)\sigma_s\}\,\mathrm{d}x\,\mathrm{d}y. \tag{64}$$

So the first variation is given by

$$\left[\frac{\mathrm{d}}{\mathrm{d}s}(\hat{H} - c\hat{m}_1)\right]_{s=0} = \int_D [\mathscr{E}_\rho(H - cm_1)\,\partial(\rho, \mu) + \mathscr{E}_\sigma(H - cm_1)\{\partial(\rho, \lambda) + \partial(\sigma, \mu)\}]\,\mathrm{d}x\,\mathrm{d}y,$$

where everything including λ and μ in the integrand is evaluated at $s = 0$ but the bar notation is now suppressed without risk of confusion. Substituting from (8) for the Euler derivatives, using the skew symmetry of the operations $\partial(\rho, .)$ and $\partial(\sigma, .)$ [see preceding footnote] and requiring the first variation to vanish, we obtain

$$0 = \int_D [\lambda\partial(\rho, \psi - cy) + \mu\{\partial(\sigma, \psi - cy) + \partial(\rho, gy - \tfrac{1}{2}|\nabla\psi|^2)\}]\,\mathrm{d}x\,\mathrm{d}y. \tag{65}$$

Since λ and μ are arbitrary continuous functions, the Du Bois–Reymond lemma of the calculus of variations therefore shows (61) to be implied by a stationary value of $\hat{H} - c\hat{m}_1$ in the specified class of variations, namely the class $\mathscr{C}$ decided by symplectic structure. The stationary property for variations determined by λ recovers the equation for ρ already noted to constitute the first component of (61). The same property respecting μ gives the second component

$$\partial(\sigma, \psi - cy) + \partial(\rho, gy - \tfrac{1}{2}|\nabla\psi|^2) = 0,$$

† For what follows it is unnecessary to impose boundary or asymptotic conditions on the component test functions λ and μ. Properties of ρ_x and ψ suffice to ensure, as is needed, that $\mathscr{E}(H - cm_1)\cdot\mathbf{J}\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\cdot\mathbf{J}\mathscr{E}(H - cm_1) \sim 0$ since it is the divergence of a vector function vanishing on horizontal plane boundaries, or as $r \to \infty$ when $D = \mathbb{R}^2$. Further regularity of the test functions would be required to define weak, distributional solutions of (61); but this aspect will not be covered here.

which has been noted to reproduce the well-known equation of Dubreil-Jacotin and Long.

This variational principle, that steady wave motions realize stationary values of $\hat{H}-c\hat{m}_1$ with c prescribed, has interest but is not the most useful characterization of such motions. Various difficulties attending it can at once be appreciated. The null solution of (61) is an extremal; at least one other x-independent solution generally exists representing a *conjugate flow* relative to the null state (B, §4.4); and solutions representing waves are found to be minimax extremals of Morse type different from those of the other two.

An alternative and prospectively much more useful characterization is that steady waves realize stationary values of $\hat{H}$ for given values of $\hat{m}_1$, in which case the wave velocity c enters as Lagrange multiplier. It appears that conditional *minima* are then mainly in question. Periodic waves with given wavelength presumably realize minima of $\hat{H}$, defined as an integral over one period, for given $\hat{m}_1$ similarly defined and for variations among periodic functions in the class $\mathscr{C}$ as has been explained. There will always be an x-independent conditional extremal for each $\hat{m}_1$, but this conjugate-flow solution will not be minimal if the prescribed period is large enough. Presumably also, solitary waves if they exist in a particular Boussinesq model realize minimal energy $\hat{H}$ for a given impulse $\hat{m}_1$, when both are defined for functions of unbounded support. A similar characterization is already known to apply to solitary and periodic water waves in a uniform channel (Benjamin 1972, Appendix C; 1974, §2); but the variational principle is simpler, at least superficially, because the Hamiltonian structure is so.

An important point regarding the interpretation of internal-wave phenomena concerns the *stability* of any steady motion shown to be a conditional minimizer in the sense described. The class $\mathscr{C}$ of variations defined by (62) covers the set of instantaneous realizations of ω in any free motion neighbouring the steady one in question, and the functionals $\hat{H}$ and $\hat{m}_1$ are constants of the motion. In other words, when according to (62) $\bar{\omega}$ is perturbed to $\omega(s) = \omega_*(x,y)$, say, which is taken as initial datum for (9) at $t=0$, the motion $\omega(x,y,t)$ for $t>0$ stays in $\mathscr{C}$ (as a comparison between (62) and (9) plainly shows) and maintains the same $\hat{H}$ and $\hat{m}_1$ as ω_*. So, if $\hat{H}(\bar{\omega})$ is a conditional minimum, the number $(\hat{H}-c\hat{m}_1)(\omega_*)-(\hat{H}-c\hat{m}_1)(\bar{\omega}) = O(s^2)$ qualifies as an invariant, non-negative Lyapunov functional, and stability in some sense is implied. This general idea in relation to hydrodynamic stability was propounded in a series of powerful papers by V. I. Arnold during the 1960s, and it had antecedents in Kelvin's writings on vortex motion.

The inference of stability associated with conditional minima is thus immediately plausible. It has to be acknowledged, however, that proof of the stability property is far from straightforward, even in simpler comparable examples. A prototype for what needs to be done can be found in papers by Benjamin (1972) and Bona (1975). As a first step towards fulfilling the Lyapunov criterion of stability, the property

$$|(\hat{H}-c\hat{m}_1)(\omega_*)-(\hat{H}-c\hat{m}_1)(\bar{\omega})| \to 0 \quad \text{as } d_1(\bar{\omega},\omega_*)\to 0 \tag{66}$$

referred to initial data is comparatively easy to establish for any reasonable metric $d_1(.,.)$. The number on the left in (66) will be $O(s^2)$ with $\omega_*\in\mathscr{C}$, and generalization to a metric neighbourhood of $\bar{\omega}$ can usually be accomplished by use of the triangle inequality for d_1 (see the cited papers).

The remaining, more difficult task is to show that for general ω the number on the left, which is a constant of the motion for $t>0$, majorizes some suitable chosen metric distance $d_2(\bar{\omega},\omega)$. Note that any translational invariant metric, such as one based on an unweighted norm, will elude the required property, because there will always be motions initially close to the steady wave that gradually diverge from it although remaining nearly the same in overall form. For example, another steady wave having phase velocity slightly different from c will do so. Therefore d_2 must be specified to measure distance between $\bar{\omega}$ and ω in some quotient class given by factorization of translations (again see the cited papers for details of this principle).

It is also difficult to prove that any non-trivial $\bar{\omega}$ realizes a conditional minimum. Analysis of the second and higher variations of $\hat{H}$ for given $\hat{m}_1$ becomes very complicated, and I have not yet succeeded in completing the work for any specific non-trivial example. For the second variation at least, the gist of the theory is nevertheless clear enough. From (62) one obtains

$$\ddot{\rho} = \partial(\dot{\rho}, \mu) + \partial(\rho, \mu_s),$$
$$\ddot{\sigma} = \partial(\dot{\rho}, \lambda) + \partial(\dot{\sigma}, \mu) + \partial(\rho, \lambda_s) + \partial(\sigma, \mu_s),$$

with everything on the right evaluated at $s = 0$ and with $\dot{\rho}$ and $\dot{\sigma}$ given by (63). The terms involving μ_s and λ_s have the same form as $\dot{\rho}$ and $\dot{\sigma}$, so they make no contribution to the conditional second variation of $\hat{H}$ as derived from (64). This second variation is in fact the integral of

$$\ddot{H} - c\ddot{m}_1 = (\psi - cy)\,\ddot{\sigma} + \dot{\psi}\dot{\sigma} - \dot{\rho}\nabla\psi\cdot\nabla\dot{\psi} + \tfrac{1}{2}\ddot{\rho}(gy - \tfrac{1}{2}|\nabla\psi|^2), \tag{67}$$

where $\dot{\psi} = \mathrm{B}_{(\rho)}\{\dot{\sigma} + \nabla\cdot(\dot{\rho}\nabla\psi)\}$, and the (isoperimetric) condition restricts λ and μ to be such that the integral of $\ddot{m}_1 = y\ddot{\sigma}$ be zero. Note that the conditional second variation of $\hat{H}$ vanishes for $\dot{\rho} = \rho_x$, $\ddot{\rho} = \rho_{xx}$, $\dot{\sigma} = \sigma_x$, $\ddot{\sigma} = \sigma_{xx}$, which case corresponds to a simple translation of $\bar{\omega}$ and is generated by $\lambda = 0$, $\mu = y$. This obvious fact is a reminder that the total variation of $\hat{H}$ can be positive definite, as needed to establish Lyapunov stability, *only* if translations are somehow excluded from competition.

Substituting for $\dot{\rho}$, $\ddot{\rho}$, $\dot{\sigma}$ and $\ddot{\sigma}$ in (67) and integrating over D, one obtains a quadratic functional of λ and μ which can be simplified marginally by use of the equations for $\bar{\rho}$ and $\bar{\sigma}$. The result is still fearsomely complex and will not be quoted in the absence of any successful application.

An outstanding candidate for exact treatment on these lines is nevertheless on offer. This is the practical example of solitary waves in liquids of great depth, best known as an attribute of the oceanic thermocline. Either the wave motion is concentrated around a top or bottom layer, across which density decreases with height but is constant far beyond, or it is concentrated around an intermediate heterogeneous layer below which density has a constant value and above which a smaller constant value. An approximate nonlinear theory of periodic and solitary waves in such systems was presented by Benjamin (1967); and solitary-wave solutions of the approximate evolutionary equation (often called the Benjamin–Ono equation; see B, §3) have been proved to be Lyapunov stable in respect of suitably tailored metrics by Bennett *et al.* (1983). Moreover, experimental observations by Davis & Acrivos (1967) have indicated that these solitary waves are highly stable, travelling over long distances without much change in form.

So there is good reason to suppose that solitary-wave solutions of the exact hydrodynamic problem should exist and be stable, although neither possibility has yet been proven. The present theory is ideally suited to this example, which exemplifies the Boussinesq model with D taken to be $\mathbb{R}^2$ or a half-space. The problem seems hard but a worthy objective for further study is indicated.

I am grateful to Mr Spencer Bowman for helpful dialogues about several aspects of this investigation. Credit is due to him for having independently discovered the facts noted in Appendix A.

Appendix A. Semi-Lagrangian equations

An alternative Hamiltonian representation of the Boussinesq model deserves to be summarized. It is superficially simpler in that the complicated operator **J** introduced in (9) and (10) is replaced by a canonical form. But it is more cumbersome in other respects and, unlike the representation studied above, it is limited essentially to motions in which the height of the isopycnic surfaces is for all x a single-valued function of density.

A semi-Lagrangian description is adopted, as used many times previously (e.g. by Benjamin 1967). The independent variables are x, t as before and η, say, which is the height of the isopycnic surfaces in the undisturbed state of the system (or η could be any parameter determining ρ uniquely). Thus $\rho = \rho(\eta)$ replaces the equation of mass conservation (3). The dependent variables are $y = y(x, \eta, t)$ and $\tau = \sigma y_\eta$, where σ is as defined by (1) although now reckoned as a function of x, η, t. In terms of the stream function ψ as before the velocity components are

$$u = \frac{\psi_\eta}{y_\eta}, \quad v = -\psi_x + \frac{y_x}{y_\eta}\psi_\eta, \tag{A 1}$$

whence it is found that

$$\tau = -\left\{\rho(1+y_x^2)\frac{\psi_\eta}{y_\eta}\right\}_\eta + \rho(2y_x\psi_{x\eta} - y_\eta\psi_{xx}) + \rho y_{xx}\psi_\eta + \rho_\eta y_x \psi_x = \mathrm{M}_{(y)}\psi, \tag{A 2}$$

say. The y-dependent linear operator $\mathrm{M}_{(y)}$ is in fact symmetric and remains strongly elliptic provided $y_\eta > 0$, as will now be assumed. Therefore, to replace (2), we have in principle

$$\psi = \mathrm{C}_{(y)}\tau, \tag{A 3}$$

and the inverse operator $\mathrm{C}_{(y)} = \mathrm{M}_{(y)}^{-1}$ incorporating the boundary conditions on ψ *qua* function of x, η is symmetric.

The Hamiltonian density is the total-energy density in the domain $D \subseteq \mathbb{R}^2$ which is now assigned the infinitesimal measure $\mathrm{d}x\,\mathrm{d}\eta = y_\eta^{-1}\,\mathrm{d}x\,\mathrm{d}y$. Thus, relative to a state of rest in which $y = \eta$ everywhere, we have

$$\begin{aligned} H &= \tfrac{1}{2}\rho(u^2+v^2)\,y_\eta + g(y-\eta)\,\rho y_\eta \\ &\sim \tfrac{1}{2}\tau\psi + g(y-\eta)\,\rho y_\eta = \tfrac{1}{2}\tau \mathrm{C}_{(y)}\tau + g(y-\eta)\,\rho y_\eta, \end{aligned} \tag{A 4}$$

where the equivalence denoted by $\sim$ is demonstrable from (A 1) and (A 2) by a straightforward calculation (cf. (7)). The Euler derivative of H with respect to τ is plainly $\mathrm{C}_{(y)}\tau = \psi$, since $\mathrm{C}_{(y)}$ is symmetric. The Euler derivative of H with respect to y can be found as the coefficient of $\dot{y}$ in the infinitesimal variation of H according to the first line of (A 4), allowance being made for equivalences and for the condition

$$0 = \dot{\tau} = -(\rho\dot{u})_\eta + y_\eta(\rho\dot{v})_x - y_x(\rho\dot{v})_\eta + \dot{y}_\eta(\rho v)_x - \dot{y}_x(rv)_\eta.$$

The two results are

$$\mathscr{E}_y H = u\tau - \{gy - \tfrac{1}{2}(u^2+v^2)\}\,y_\eta, \quad \mathscr{E}_\tau H = \psi. \tag{A 5}$$

(A term $g(\eta\rho)_\eta$ is omitted from $\mathscr{E}_y H$, being a function of η alone and so having no dynamical significance.)

From (A 5) the dynamical problem is seen to be represented by the equations

$$y_t = -(\mathscr{E}_\tau H)_x, \quad \tau_t = -(\mathscr{E}_y H)_x. \tag{A 6}$$

The first equation is kinematic, following immediately from (3) and the facts that $\rho(x,y,t) = \rho(\eta)$ and $\partial(x,y)/\partial(x,\eta) = y_\eta > 0$. Note that its right side is $-\psi_x = v - uy_x$. The second equation is found to be equivalent to (4) expressed in the new variables. These equations exemplify the Hamiltonian form (9) but with a simpler cosympletic operator $\mathbf{J}$, namely the 2×2 matrix with zero diagonal elements and off-diagonal elements $-\partial_x$. Hence upon the introduction of ϕ such that $\phi_x = \tau$, (A 6) may be reduced formally to Hamilton's equations (i.e. $y_t = \mathscr{E}_\phi H$, $\phi_t = -\mathscr{E}_y H$).

The Hamiltonian system (A 6) is essentially equivalent to (9) and appears to embody no new information. Its symmetries correspond exactly to those of (9) and its associated conservation laws are just the transpositions of those for (9) into the new variables. It is worth noting the modified forms of the first six symmetries, which underlie conservation laws. Let us again write $\boldsymbol{\omega}$ for the solution, now the column vector $[y,\tau]^{\mathrm{T}}$, and $\boldsymbol{P}$ for the standard representative of the infinitesimal generator of a one-parameter symmetry subgroup. Corresponding to the first part of Theorem 1, we now have $\boldsymbol{P}_3 = -\boldsymbol{\omega}_t$, $\boldsymbol{P}_4 = -\boldsymbol{\omega}_x$, $\boldsymbol{P}_5 = [1,0]^{\mathrm{T}}$, $\boldsymbol{P}_6 = t\boldsymbol{\omega}_x + [0,\rho_\eta]^{\mathrm{T}}$, $\boldsymbol{P}_7 = [t, -\rho_\eta y_x]^{\mathrm{T}}$, $\boldsymbol{P}_8 = (y+\frac{1}{2}gt^2)\,\boldsymbol{\omega}_x + [x, gt\rho_\eta + \tau y_x]^{\mathrm{T}}$. As regards conserved densities, integrable over an unbounded D in the case of a localized wave motion, the first two given in Theorem 4 are evidently replaced by $y-\eta$ and τ. The third is H as expressed by (A 4), and the remaining five are the obviously transposed forms of those in Theorem 4.

Appendix B. Confirmation of Hamiltonian structure

Although the proven property (18) of the operator $\mathbf{J}$ given by (10) attests basically to the Hamiltonian structure of the Boussinesq system (9), it is desirable to summarize a verification of the closure condition as explained by Olver (1980*b*) in the language of differential forms. Appeal can be made to an important result of his which simplifies the issue (Olver 1980*b*, lemma 4.5), and the calculations needed at present are comparable with those used by him to confirm one of the several Hamiltonian structures attributable to the dynamical equations for a homogeneous perfect fluid (Olver 1982, §3). The essential points are noted as follows without full explanation, for which reference may be made to Olver's papers.

By analogy with canonical Hamiltonian systems, the required condition is that the fundamental symplectic two-form

$$\Omega = -\tfrac{1}{2}\,\mathrm{d}\boldsymbol{\omega}^{\mathrm{T}} \wedge \mathbf{J}^{-1}\,\mathrm{d}\boldsymbol{\omega}$$

should be closed. In our problem the formal inverse $\mathbf{J}^{-1}$ of the matrix of differential operators $\mathbf{J}$ is not easily defined, however, and so attention is given, rather, to the associated cosymplectic two-form

$$\begin{aligned}\tilde{\Omega} &= \tfrac{1}{2}\,\mathrm{d}\boldsymbol{\omega}^{\mathrm{T}} \wedge \mathbf{J}\,\mathrm{d}\boldsymbol{\omega} = \tfrac{1}{2}[\mathrm{d}\rho, \mathrm{d}\sigma] \wedge \mathbf{J}[\mathrm{d}\rho, \mathrm{d}\sigma]^{\mathrm{T}} \\ &\sim \rho\,\mathrm{d}\rho_x \wedge \mathrm{d}\sigma_y - \rho\,\mathrm{d}\rho_y \wedge \mathrm{d}\sigma_x + \sigma\,\mathrm{d}\sigma_x \wedge \mathrm{d}\sigma_y.\end{aligned} \tag{A 8}$$

(Here as before the equivalence $\sim$ means equality except for a divergence.) According to Olver's lemma 4.5, $\mathbf{J}$ is Hamiltonian if and only if

$$\mathrm{d}_{\mathbf{J}}(\tilde{\Omega}) \sim 0 \tag{A 9}$$

in the space of three-forms. Here $\mathrm{d_J}$, the modified exterior derivative based on $\mathbf{J}$, is a linear operation with the properties $\mathrm{d_J}\omega = \mathbf{J}\,\mathrm{d}\omega$, i.e.

$$\left.\begin{aligned} \mathrm{d_J}\rho &= -\rho_x\,\mathrm{d}\sigma_y + \rho_y\,\mathrm{d}\sigma_x, \\ \mathrm{d_J}\sigma &= -\rho_x\,\mathrm{d}\rho_y + \rho_y\,\mathrm{d}\rho_x - \sigma_x\,\mathrm{d}\sigma_y + \sigma_y\,\mathrm{d}\sigma_x, \end{aligned}\right\} \tag{A 10}$$

also $\mathrm{d_J}\cdot\mathrm{d} = 0$ and the derivation property on forms, giving for example

$$\mathrm{d_J}(\rho\,\mathrm{d}\rho_x \wedge \mathrm{d}\sigma_y) = (\mathrm{d_J}\rho) \wedge \mathrm{d}\rho_x \wedge \mathrm{d}\rho_y.$$

Written in full, the result from (A 8) and (A 10) is

$$\begin{aligned} \mathrm{d_J}(\tilde{\Omega}) \sim{} & (\rho_x\,\mathrm{d}\sigma_y - \rho_y\,\mathrm{d}\sigma_x) \wedge (\mathrm{d}\rho_y \wedge \mathrm{d}\sigma_x - \mathrm{d}\rho_x \wedge \mathrm{d}\sigma_y) \\ & - (\rho_x\,\mathrm{d}\rho_y - \rho_y\,\mathrm{d}\rho_x + \sigma_x\,\mathrm{d}\sigma_y - \sigma_y\,\mathrm{d}\sigma_x) \wedge (\mathrm{d}\sigma_x \wedge \mathrm{d}\sigma_y), \end{aligned}$$

in which four of the eight components are null because of duplicated entries in the triple cross-product. There remains

$$\begin{aligned} \mathrm{d_J}(\tilde{\Omega}) \sim{} & \rho_x(\mathrm{d}\sigma_y \wedge \mathrm{d}\rho_y \wedge \mathrm{d}\sigma_x - \mathrm{d}\rho_y \wedge \mathrm{d}\sigma_x \wedge \mathrm{d}\sigma_y) \\ & + \rho_y(\mathrm{d}\sigma_x \wedge \mathrm{d}\rho_x \wedge \mathrm{d}\sigma_y + \mathrm{d}\rho_x \wedge \mathrm{d}\sigma_x \wedge \mathrm{d}\sigma_y) = 0. \end{aligned}$$

Thus the closure condition is confirmed.

REFERENCES

Abarbanel, H. D. I., Holm, D. D., Marsden, J. E. & Ratiu, T. 1986 Nonlinear stability analysis of stratified fluid equilibria. *Phil. Trans. R. Soc. Lond.* A (to appear).

Arnold, V. I. 1978 *Mathematical Methods of Classical Mechanics.* Springer.

Benjamin, T. B. 1966 Internal waves of finite amplitude and permanent form. *J. Fluid Mech.* **25**, 241–270.

Benjamin, T. B. 1967 Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.* **29**, 559–592.

Benjamin, T. B. 1972 The stability of solitary waves. *Proc. R. Soc. Lond.* A **328**, 153–183.

Benjamin, T. B. 1974 Lectures on nonlinear wave motion. *Amer. Math. Soc., Lectures in Appl. Math.* **15**, 3–47.

Benjamin, T. B. 1984 Impulse, flow force and variational principles. *IMA J. Appl. Math.* **32**, 3–68.

Benjamin, T. B. & Olver, P. J. 1982 Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125**, 137–185.

Bennett, D. P., Brown, R. W., Stansfield, S. E., Stroughair, J. D. & Bona, J. L. 1983 The stability of internal solitary waves. *Math. Proc. Camb. Phil. Soc.* **94**, 351–379.

Bona, J. L. 1975 On the stability of solitary waves. *Proc. R. Soc. Lond.* A **344**, 363–374.

Davis, R. E. & Acrivos, A. 1967 Solitary internal waves in deep water. *J. Fluid Mech.* **29**, 593–607.

Dubreil-Jacoton, M. L. 1935 Complément à une note antérieure sur les ondes de type permanent dans les liquides hétérogènes. *Atti. Accad. noz. Lincei. Rc.* (6) **21**, 344–346.

Hardy, G. H., Littlewood, J. E. & Pólya, G. 1964 *Inequalities*, 2nd edn. Cambridge University Press.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Long, R. R. 1953 Some aspects of the flow of stratified fluids. I. A theoretical investigation. *Tellus*, **5**, 42–57.

Manin, Yu. I. 1979 Algebraic aspects of nonlinear differential equations. *J. Soviet Math.* **11**, 1–122.

Marsden, J. E. 1976 Well-posedness of the equations of a non-homogeneous perfect fluid. *Comm. PDE* **1**, 215–230.

Olver, P. J. 1979 How to find the symmetry group of a differential equation. Appendix in D. H. Sattinger, *Group Theoretic Methods in Bifurcation Theory.* Lecture Notes in Math., Vol. 762, pp. 200–239. Springer.

OLVER, P. J. 1980*a* *Applications of Lie Groups to Differential Equations.* Mathematical Institute, University of Oxford, Lecture Notes.

OLVER, P. J. 1980*b* On the Hamiltonian structure of evolution equations. *Math. Proc. Camb. Phil. Soc.* **88**, 71–88.

OLVER, P. J. 1982 A nonlinear Hamiltonian structure for the Euler equations. *J. Math. Anal. Applics.* **89**, 233–250.

OLVER, P. J. 1983 Conservation laws of free boundary problems and the classification of conservation laws for water waves. *Trans. Am. Math. Soc.* **277**, 353–380.

RIPA, P. 1981 Symmetries and conservation laws for internal gravity waves. *Proc. Am. Inst. Phys.* **7b**, 281–306.

SELIGER, R. L. & WHITHAM, G. B. 1968 Variational principles in continuum mechanics. *Proc. R. Soc. Lond.* A **305**, 1–25.

YIH, C.-S. 1965 *Dynamics of Nonhomogeneous Fluids.* Macmillan.